EXERCISES

PION | 2019

5E LUSTRUM

UNIVERSITY OF TWENTE.

nederlandse natuurkundige vereniging

TU/e APPLIED PHYSICS

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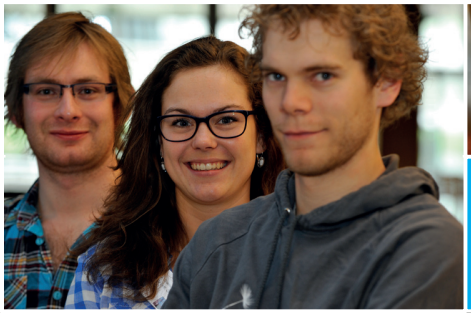

Gezicht van de natuurkunde

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Preface

Dear PION Participant,

PIONmeunig mooi that you're participating in this year's PION edition! We proudly present to you the exercises of PION2019. The committee has been busy the last couple of months to assemble this set of physics problems to challenge you. The problems cover a whole range of physics subjects, from fluid physics up to quantum computing. If your team is capable enough to solve these problems well enough, you can participate in this year's *Physics League Across Numerous Countries for Kick-ass Students* in Denmark. The committee wishes your team good luck, but most of all lots of fun!

From left to right: Frank Somhorst (problems), Aron van den Bogaard (secretary), Thies Jansen (chairman), Jardi Timmerhuis (treasurer), Lars Bossink (external affairs).

Rules

- **Please make every exercise on a separate sheet of paper stating your team's name! Please answer all questions in English.**
- It is not allowed to use any (study)books, notes, etc. other than BINAS and a dictionary.
- It is not allowed to communicate (about the problems) via any means with anyone other than team members.
- It is allowed to use a scientific, graphical calculator (Ti-83, Ti-84 or a similar model).
- Mobile phones are tolerated, but can only be used for emergencies, during the duration of the competition.
- The winner of PION2019 is the best team, with the most points. Three teams will get a ticket to PLANCKS: the best bachelor team and the two best teams that are not the best bachelor team.

In cases where these rules do not provide, the PION2019-committee will decide.

Contents

1 Graphene with a twist

Prof. Dr. Ir. H.J.W. Zandvliet, Dr. K. Sotthewes, Dr. A. Van Houselt, Dr. E.S. Kooij
Physics of Interfaces and Nanomaterials - University of Twente
10 points

Graphene is a two-dimensional material consisting of sp^2 hybridized carbon atoms that are arranged in a honeycomb lattice (see Figure 2). The distance between nearest neighbour carbon atoms is a; the lattice constant amounts to $a\sqrt{3}$.

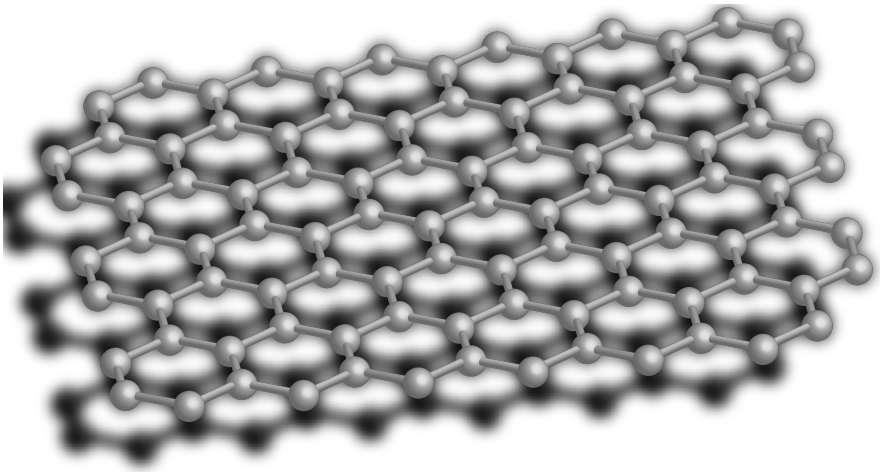

Figure 2: A single layer of graphene, in which carbon atoms are arranged in a honeycomb lattice.

The unit cell of graphene contains two carbon atoms. The reciprocal lattice can be constructed using the relation $\exp(i\mathbf{G}\cdot\mathbf{R}) = 1$, where $\mathbf{R}=\mathbf{a}_1+\mathbf{a}_2$ spans the lattice in real space and $\mathbf{G}=\mathbf{b}_1+\mathbf{b}_2$ the lattice in reciprocal space.

a) (2 pts) Determine the reciprocal lattice of graphene. Define lattice vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ of the real lattice and determine length and direction of the reciprocal vectors $\mathbf{b}_1$ and $\mathbf{b}_2$.

To determine the scattering amplitude and phase of a reflected plane wave, the structure factor is very helpful; it is given by $S = \sum_j f_j \exp(-i\mathbf{b}_j \cdot \mathbf{r}_j)$, where $\mathbf{r}_j$ represents the relative position vector of the atoms in the unit cell and f_j the atomic form factor of the carbon atoms.

b) (2 pts) Determine the structure factor of the unit cell of graphene.

Consider two graphene layers stacked in the so-called Bernal configuration; half of the carbon atoms in the top layer are located exactly above a carbon atom of the bottom layer, whereas the other half are located above the center of the honeycomb cells of the bottom layer). When the top layer is rotated by a small twist angle θ with respect to the bottom layer, twisted bilayer graphene is formed. This structure exhibits a so-called Moiré pattern with periodicity L, which is much larger than the periodicity of a single layer of graphene, as shown in Figure 3.

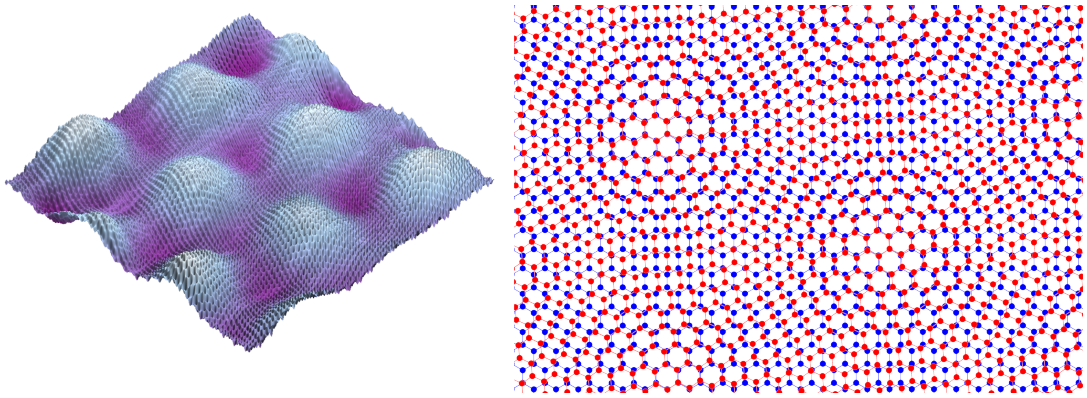

Figure 3: (left) Scanning tunneling microscopy (STM) image of twisted graphene. (right) Schematic Moiré pattern created by rotating two graphene layers (red and blue) relative to each other.

c) (2 pts) Schematically plot the variation of the periodicity L as a function of the twist angle θ.

d) (2 pts) Derive an analytical expression for the periodicity L as a function of the twist angle θ.

The dispersion relation for electrons in graphene near the Fermi level is given by $E = \hbar v_{\mathrm{F}} |\mathbf{k}|$, where $\hbar$ is the reduced Planck constant, v_{F} the Fermi velocity and $\mathbf{k}$ the momentum of the electrons. The Fermi level can shift due to doping, as schematically shown in Figure 4.

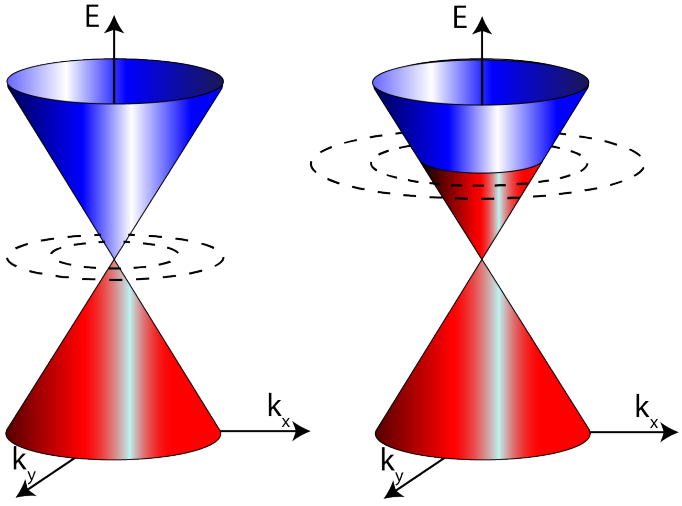

Figure 4: Electron dispersion relation near the Fermi level for undoped (left) and doped (right) graphene.

e) (2 pts) Derive an analytical expression for the charge density in graphene in the case that the Fermi level lies at an energy E_{F} above the Dirac point, also referred to as the charge neutrality point.

2 Three-level quantum system

Dr. J.W.J. Verschuur
Staff Applied Physics - University of Twente
10 points

Consider a three-level quantum system described by a Hermitian Hamiltonian, that can be written as the sum of two parts:

$$H = H_0 + \lambda H_1$$

where λ is a real number. The eigenstates of H_0 are $|1\rangle$, $|2\rangle$ and $|3\rangle$, with corresponding eigen values, given by:

$$\begin{aligned} H_0 |1\rangle &= 0 \\ H_0 |2\rangle &= \alpha |2\rangle \\ H_0 |3\rangle &= \alpha |3\rangle \end{aligned}$$

This three-level system can be described in matrix notation. The idea is that you obtain a full description of the system from the characteristics given below.

a) (1 pt) Write down H_0 in matrix notation in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis.

b) (2 pts) Write down H_1 in matrix notation, using the lowest amount of unknown different matrix elements (in the same basis). Use symmetry considerations to limit the number of elements in the matrix without losing generality.

When the (eigenvalue) spectrum of H is computed using perturbation theory, it is found that the eigenstates of H to lowest order in λ are:

$$\begin{aligned} |a\rangle &= |1\rangle \\ |b\rangle &= \frac{1}{\sqrt{2}} \left(|2\rangle + |3\rangle \right) \\ |c\rangle &= \frac{1}{\sqrt{2}} \left(|2\rangle - |3\rangle \right) \end{aligned}$$

and that the corresponding eigenvalues are:

$$\begin{aligned} E_a &= -\frac{\lambda^2}{\alpha} + \mathcal{O}(\lambda^3), \\ E_b &= \alpha + \lambda + \frac{\lambda^2}{\alpha} + \mathcal{O}(\lambda^3), \\ E_c &= \alpha - \lambda + \mathcal{O}(\lambda^3). \end{aligned}$$

c) (4 pts) Determine as many of the matrix elements of H_1 as you can.

d) (2 pts) Two different bases are used in this problem, the numeric basis and the alpha-numeric basis. Give the transformation, that relates the two bases.

e) (2 pts) Use the transformation to represent the Hamiltonian H in both representations.

f) (1 pt) Is either one of the two bases an eigen-system for the Hamiltonian H?

Perturbation theory

For your convenience, the expressions to calculate the corrections in first and second order respectively, are given by:

$$\begin{aligned} E_i^{(1)} &= \langle i|H_1|i\rangle \\ E_i^{(2)} &= \sum_{j\neq i}^{N} \frac{\left|\langle j|H_1|i\rangle\right|^2}{E_i^0 - E_j^0} \end{aligned}$$

The first-order correction to the wave function is given by:

$$|i\rangle^{(1)} = \sum_{j\neq i}^{N} \langle j|H_1|i\rangle \, |j\rangle$$

Note that these expression only hold for non-degenerate perturbation theory. Degenerate perturbation, if applicable, is part of the problem.

3 Putting equilibrium fluctuations to work

Dr. K.H. Mathwig and Prof. Dr. S.J.G. Lemay
Bio Electronics - University of Twente
10 points

In nanofluidics, a fluid is confined to sub-micron length scales in one or more dimension. This can cause effects which are normally too small to be noticed to become dominant. Figure 5 shows a sketch of a lithographically fabricated nanofluidic device. It consists of a nanochannel of height ∼100 nm and width 5 μm. Electrode pairs of length $L = 50$ μm are embedded in its floor and ceiling, which allows exchanging electrons with molecules in the fluid via electrochemical reactions. This can be employed to create extremely sensitive detectors via so-called redox cycling. In this method, potentials are applied to the electrodes such that the floor of the channel can donate electrons to electrochemically active molecules that come within the electron tunneling distance (∼1 nm) while the ceiling can similarly accept electrons. Molecules undergoing Brownian motion in the nanochannel repeatedly collide with the electrodes, ferrying electrons from floor to ceiling and thus generating a measurable electrical current. Because of the small distance between the electrodes, this process is very efficient and a single molecule ferries about 10^5 electrons/second.

Here we consider a simple model of mass transport in this system. Because of confinement in the width and height directions, we only consider a single dimension, x, the position along the length of the channel. The measured current is proportional to the number of molecules present in the detection volume between the electrodes ($0 < x < L$); for simplicity we take this number $N(t)$ to represent the measured signal. Molecules can either diffuse or undergo convection along the channel such that the local number density of particles at position x, $n(x,t)$ (with units of particles per unit length), obeys the drift-diffusion equation,,

$$\frac{\partial n(x,t)}{\partial t} = D\frac{\partial^2 n(x,t)}{\partial x^2} - v\frac{\partial n(x,t)}{\partial x}. \tag{1}$$

Here D is the diffusion coefficient for the molecules (a constant) and v is the advection velocity of the fluid in the x direction. We do not worry about the inlet or outlet regions, considering the channel to be infinite in the $-x$ and $+x$ directions.

The average number of molecules, $\bar{N}$, is simply given by the concentration of molecules in the fluid times the volume of the detection region between the electrodes. However, molecules can enter and leave the detection volume by Brownian motion. Since molecules diffuse independently of each other, this causes $N(t)$ to fluctuate randomly in time around $\bar{N}$, conveniently written as $N(t) = \bar{N} + \Delta N(t)$. Such fluctuations are universal for systems in diffusive equilibrium, but usually they are too small to be detected (more precisely, $\Delta N/\bar{N}$ vanishes in the thermodynamic limit). Because the present system is very small, however (volume 25 femtoliters), $\bar{N}$ is small and the fluctuations $\Delta N(t)$ become observable. In fact, these fluctuations are the dominant source of noise in sensors based on this device, as illustrated in Figure 5(b). Here we look further into these fluctuations based on the 1D model introduced above.

a) (2 pts) First consider a single pair of facing electrodes. We first focus on the purely diffusive case ($v = 0$). Suppose that a single molecule starts at $t = 0$ somewhere in the detection region, $0 < x < L$. As time passes, the chance that the molecule is still in the detection region diminishes. Give an expression for the characteristic time scale over which this process occurs. Justify your answer in 1-3 sentences and equations/diagrams as needed.

b) (4 pts) More quantitatively, we define the autocorrelation function,

$$C_{11}(\tau) = \left\langle \Delta N(t) \Delta N(t+\tau) \right\rangle, \tag{2}$$

where the brackets represent an average over time. Evaluate $C_{11}(\tau)$ for this system. We are only interested in the τ dependence; you need not normalize your result.

c) (1 pt) Suppose that we now have two detectors, each of length L, localized along the length of the nanochannel and separated by a gap of lenth g. Gap 1 is upstream and gap 2 is downstream. In the presence of a finite advection velocity $v > 0$, number fluctuations at the upstream detector can propagate to the downstream detector. If this takes too long, however, Brownian motion will wash out the fluctuations before they can reach the second detector. Assuming that $g \gg L$, give an estimate of the minimum characteristic velocity v required for fluctuations to be preserved between detectors. Justify your answer in 1-3 sentences and equations/diagrams as needed.

d) (3 pts) We now define the crosscorrelation function,

$$C_{12}(\tau) = \left\langle \Delta N_1(t) \Delta N_2(t+\tau) \right\rangle, \tag{3}$$

where $\Delta N_1(t)$ and $\Delta N_2(t)$ are the simultaneously measured fluctuations at the upstream and downstream detectors, respectively. Derive an expression for $C_{12}(\tau)$, again disregarding normalization.

The cross-correlation function can be used to infer the advection velocity of the fluid in the nanochannel, as illustrated in Figure 6 for the case $g \ll L$. This is not a trivial achievement, as flow rates in these nanochannels are only a few drops per year! Figure 6(c) is the result that you are asked to derive in part (d).

This question is based on K. Mathwig, D. Mampallil, S. Kang and S. G. Lemay, Phys. Rev. Lett. **109**, 118302 (2012).

Some useful mathematical details:

1. The solution to Equation 1 for a single particle starting at the origin ($n(x, 0) = \delta(x)$) is

$$n(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-x^2/4Dt}, \tag{4}$$

2. You may find it convenient to use the shorthand notation for the error function,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du, \tag{5}$$

where $\mathrm{erf}(0) = 0$, $\mathrm{erf}(\infty) = 1$, $\mathrm{erf}(-x) = -\mathrm{erf}(x)$ and

$$\int \mathrm{erf}(x)dx = x\,\mathrm{erf}\,x + \frac{e^{-x^2}}{\sqrt{\pi}} \tag{6}$$

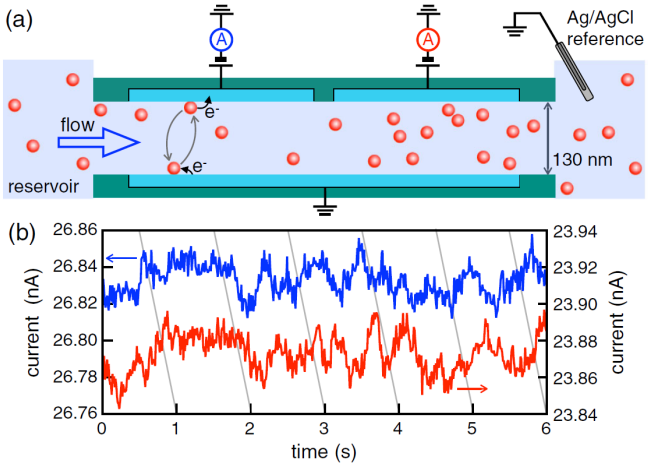

Figure 5: (a) Schematic of the concept for measuring fluctuations in the number of electrochemically active molecules ($N(t)$, red balls) between two electrodes in a nanochannel. (b) Raw current-time traces recorded at two electrode pairs located downstream from each other. The noise is due to statistical fluctuations in $N(t)$. These data were obtained at a flow rate corresponding to a transit time between the electrode pairs of approximately 0.2 s. Gray lines are guides to the eye to highlight correlations between the two traces.

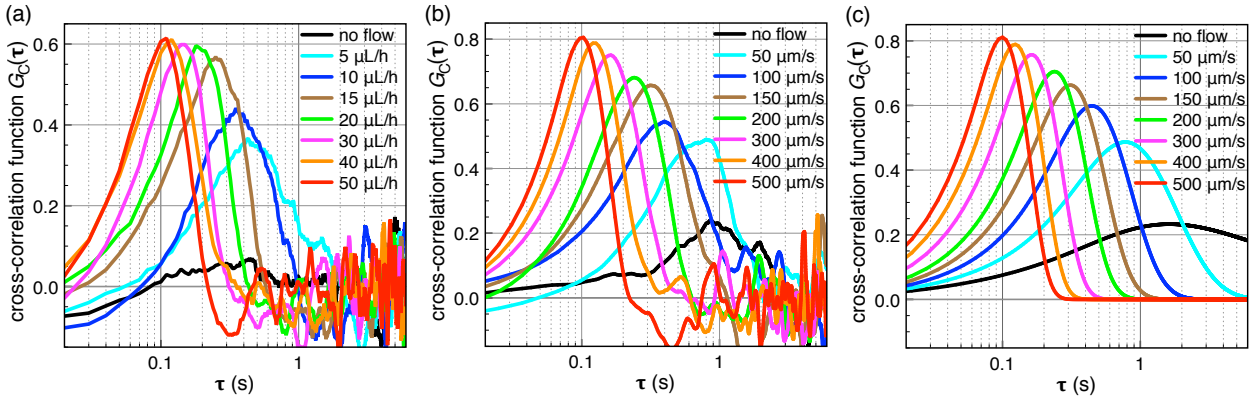

Figure 6: (a) Cross-correlation functions of current-time traces recorded experimentally at the two electrode pairs ($L = 50$ μm, $g = 2$ μm) for different pump flow rates. (b) Cross-correlation functions obtained from a one-dimensional random walk simulation of Equation (1). The noise in the traces at large τ is comparable to that in panel (a) because the simulation was performed over the same time interval. (c) Analytically derived cross-correlation function. This is a plot of the answer to part (d) of this question.

4 Infinite potential

Dr. Ir. H.L. Offerhaus
Optical Sciences - University of Twente
10 points

Estimate the magnetic vector potential $\mathbf{A}$ inside a cylinder of radius $R = 5mm$ and length $L = 50mm$ that has a current I flowing longitudinally over the surface which is uniformly distributed over the circumference. As $R << L$, the cylinder can be considered to be thin and long.

Hint: Recall that $\mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}}{r}\, dV$.

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5 Touching pulses

Dr. Ir. H.L. Offerhaus
Optical Sciences - University of Twente
10 points

Given is the following expression for intramodal dispersion of refractive index n_1 for a single mode through a fiber as:

$$\lambda^2 \frac{d^2 n_1}{d\lambda^2} = 0.025$$

and a laser that produces pulses of 20 fs duration ($1/e$ width) centered around a wavelength of 850 nm with a repetition frequency of 100 MHz.
How many km's of fiber do you need to let the tail ($1/e$ point) of one pulse be touched by the front ($1/e$) of the next pulse?

Hint: Recall that the group velocity v_g is given by $v_g = \frac{c}{n_1 - \lambda \frac{dn_1}{d\lambda}}$.

6 The Clements decomposition

Dr. J.J. Renema
Complex Photonic Systems - University of Twente
10 points

In quantum optics, we are quite frequently interested in making a set of optical modes interfere in a controlled fashion. Such control is particularly important when performing quantum computations with optical systems, since the way the optical modes interfere controls the computation which is being performed. One way to do this is in photonic chips, where a series of waveguides is written in some dielectric medium. Light is then typically sent into the side of the chip via optical fibers, and the chip then functions as a kind of shunting yard for light.

In this exercise, we will explore some of the quantum mechanics of the process of programming such a chip. In particular, we are interested in how many resources (optical elements) are required to produce controlled interference of N distinct optical modes.
If an optical system is lossless, the transformation on the optical modes is given by a unitary matrix U, in the form of $E_{out} = UE_{in}$, where E is the vector of electric field amplitudes.

a) (2 pts) Explain why the fact that we require the system to be lossless means that the number of input modes has to be equal to the number of output modes, i.e. why U has to be a square matrix. *Hint: keep in mind that linear optical systems must obey reciprocity.*

Recall that a unitary matrix is defined by the relation $U^\dagger U = I$, i.e. $(U_i, U_j) = \delta_{ij}$, where $(,)$ is the inner product and U_i is the i-th column of U.

b) (2 pts) Explain why the lossless character of the transformation implies that U must not just be square, but also unitary.

We would like to have the power to program out arbitrary unitary optical transformations. Surprisingly, this can be done with a relatively modest number of optical components. To guess how many components we need, we will count the number of free parameters in a unitary transformation.

c) (2 pts) Compute the number of real free parameters in a unitary matrix. Hint: begin with an arbitrary complex matrix, and consider the number of restrictions which unitarity imposes. Furthermore, consider that since we will not re-interfere the light after the unitary transformation, the phase of the light at the output of the system doesn't matter, which further reduces the number of free parameters by N.

The elementary component which we have at our disposal is an arbitrary 2-by-2 optical transformation, composed of a single beam splitter and a phase shifter.

d) (2 pts) Show that the number of such components required is $N(N-1)/2$

e) (1 pt) What physical interpretation can you give to this number?

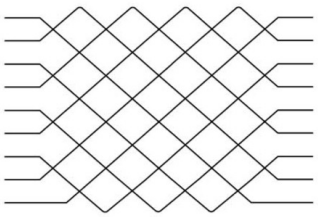

Figure 7: The Clements decomposition. Each line represents an optical mode, and each crossing point between two modes represents an arbitrary 2-mode transformation.

That this number of beam splitters and phase shifters is indeed sufficient to implement an arbitrary optical transformation was rigorously proven by (then PhD student) William Clements in 2016. His scheme has the particularly attractive feature that it only requires transformations between adjacent modes, which means that it can be implemented in a planar chip, i.e. without optical modes going out of plane, which significantly simplifies fabrication. A picture of the scheme is given in Figure 7.

f) (1 pt) Give a simple physical argument why deleting a single row of beam splitters from this scheme would make it no longer universal.

7 Fast spinning with electrons

Dr. Ir. M. Veldhorst and Ir. N. Hendrickx
QuTech - Delft University of Technology
15 points

In quantum information it is crucial that information can be stored on quantum bits (qubits) for a long time. In quantum computation with silicon quantum dots, single electrons are confined with electric gates and the spin states are used to define qubits. An important time scale is the decay time over which an electron in the spin up state relaxes to the spin down state. This time can be measured by preparing a spin down electron, applying a π-pulse using electron-spin-resonance (ESR) such that the electron spin evolves to the spin up state, and by measuring the probability of the electron spin being in the spin up state as a function of waiting time. The measurement can be done using a step called spin-to-charge conversion, see the drawing below, where nearby charge sensors are used to detect tiny changes in the capacitance.

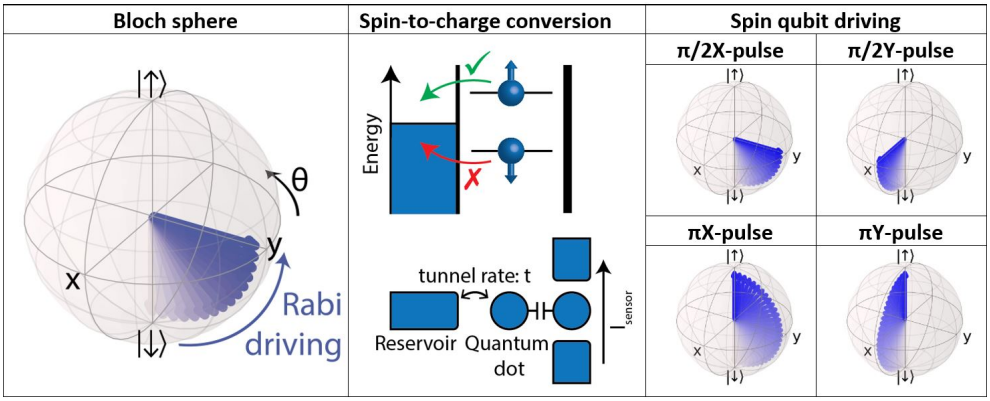

Figure 8: The spin qubit quantum toolbox.

Readout

a) (1 pt) Draw the signal of the charge sensor when a quantum dot is filled with an electron in the spin down state and tuned to the readout position. Include labels associated with relevant time scales.

b) (2 pts) Draw the signal of the charge sensor when a quantum dot is filled with an electron in the spin up state and tuned to the readout position. Include labels associated with relevant time scales.

c) (1 pt) Explain how readout leads to spin initialization and give the initialized spin.

Qubit operation

The qubit state can be changed by applying well calibrated ESR pulses, such that the spin rotates over a certain angle around a certain axis. Four different pulses that can be applied are given in Figure 8.

d) (1 pt) How can we turn off spin-to-charge conversion during these operations?

e) (2 pts) Describe how the experiment to measure the spin life time needs to be modified in order to measure the dephasing time of a single spin.

A silicon spin qubit is prepared pointing along the positive x-axis direction. In practical experiments, qubits are subject to several types of noise. Consider constant magnetic noise, such that the actual field is 1 nT lower than targeted. Due to the slower Larmor precession, the qubit acquires a phase shift over time as compared to the targeted precession.

f) (1 pt) Show how one can calculate the acquired phase shift over time.

Assume for the following two questions that driving takes zero time and thus time needs to be spend precessing.

g) (3 pts) Design a sequence consisting of a single pulse for a given sequence length τ such that the qubit (prepared pointing in the positive x-axis direction) has acquired zero phase shift compared to the situation if there was no constant noise. Write down the sequence and include the length of each segment. Also draw on the Bloch sphere the spin evolution due to the pulses and the noise.

h) (2 pts) Now assume the qubit is initialized at $\theta = \frac{\pi}{4}$ from the x-axis. Give the simplest pulse sequence for a given sequence length τ such that the qubit has acquired zero phase shift compared to the situation if there was no noise. Draw on the Bloch sphere the spin evolution due to the pulses and the noise.

Instead of a constant magnetic noise term, the spin qubit is now subject to a completely randomly fluctuating magnetic field.

i) (1 pt) Can you still define a pulse sequence such that for a given time τ the qubit has acquired zero phase shift compared to its initial state? Explain why.

In experiments, one encounters often a noise spectrum with a 1/f spectral density, which is believed to originate from defects in the substrate close to the spin qubits. Quantum operations in experiments take a finite time and we thus strive to maximize the number of operations within the dephasing time to do meaningful quantum calculations.

j) (1 pt) If we can find a method to double the speed at which we can do qubit operations, can we do more, less, or exactly two times as many rotations? Explain why.

8 Pointy ice drops

Prof. Dr. J.H. Snoeijer, Dr. A. Marin, Dr. S.G. Huisman
Physics of Fluids - University of Twente
10 points

When a drop of liquid water is placed on a cold surface, it freezes into a singular shape. The initially rounded liquid drop is transformed into a pointy ice drop, which at its tip exhibits a conical shape. This freezing process can be seen in the photographs of Figure 9(ab). The water drops are initially at room temperature, while the surface is at a temperature that can range from $-10°$C to $-40°$C. The key ingredient behind this phenomenon is that the density of solid ice (ρ_s) is lower than the density of liquid water (ρ_ℓ). We therefore introduce the density ratio $\nu = \rho_s/\rho_\ell$.

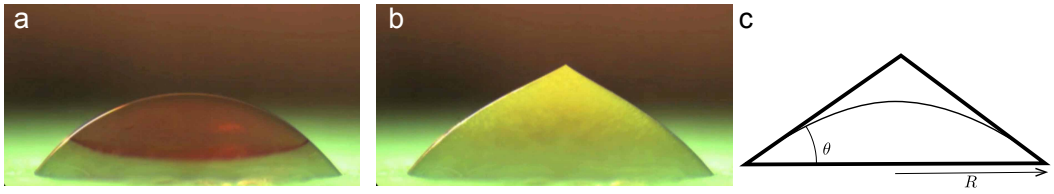

Figure 9: (a) Photograph of a partially frozen drop, after it is placed onto a cold surface. The bottom part of the drop has turned into solid ice (green), while the top part is still liquid (red). (b) Photograph of the fully solidified drop, exhibiting a conical tip. (c) The tip formation is modelled by a transformation from a liquid "spherical cap" to a solid "cone". Both the cap and the cone have the same base radius R and the same angle θ.

As a first model, we assume that the unfrozen liquid mass in Figure 9(a) can be approximated by a perfect spherical cap with a flat base, with base radius R and angle θ – see Figure 9(c) for a definition. The volume of a spherical cap reads $V_{\text{cap}} = R^3 f(\theta)$, with

$$f(\theta) = \frac{\pi}{3}\left(\frac{2 - 3\cos\theta + \cos^3\theta}{\sin^3\theta}\right). \tag{7}$$

The spherically-shaped liquid transforms into a cone of ice, with the same base radius R and the same angle θ (Figure 9(c)).

a) (1 pt) Show that the volume of the cone $V_{\text{cone}} = \frac{\pi}{3}\tan(\theta)R^3$.

An equation for the angle θ is obtained by comparing the liquid mass stored in the spherical cap to the frozen mass of the final ice cone.

b) (2 pts) Introducing $x = \cos\theta$, show that mass conservation implies:

$$2x - 3x^2 + x^4 - \nu\left(1 - x^2\right)^2 = 0. \tag{8}$$

The physically relevant solution to this equation reads

$$x = \frac{1}{\sqrt{1-\nu}} - 1. \tag{9}$$

c) (2 pts) Determine the range of ν for which this model predicts a conical solution.

The model is able to explain the formation of a cone, but unfortunately it does not capture the phenomenon for the density ratio of water $\nu \approx 0.92$. The flaw of the model based on Figure 9(c) is the implicit assumption that the freezing front is perfectly flat. Figure 10 provides a more accurate view of the freezing front, which has two important geometric features: (i) the front has a spherical shape, (ii) the front ends perpendicularly to the ice-air interface.

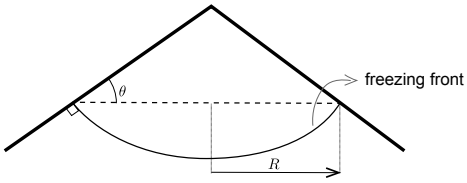

Figure 10: Sketch of the refined model. The freezing front has a (downward) spherical shape, that ends perpendicularly to the ice-air interface.

d) (2 pts) Based on these refined assumptions, derive the mass conservation equation.

e) (1 pt) Using a graphical calculator, determine the angle θ for $\nu \approx 0.92$.

f) (2 pts) Provide a physical motivation why the freezing front is perpendicular to the ice-air interface.

9 The relativity of electric and magnetic fields

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10 points

Consider a negative charge $q(-)$ moving with velocity $v_0 = v$ in frame **S** parallel to a current-carrying wire as in Figure 11. The density of the (negative) conduction electrons is ρ_- and their velocity is $v_- = v$. The density of the fixed, positive nuclear charges is ρ_+ with velocity $v_+ = 0$ in frame **S**. We consider an uncharged wire, so $\rho_+ = \rho_-$.

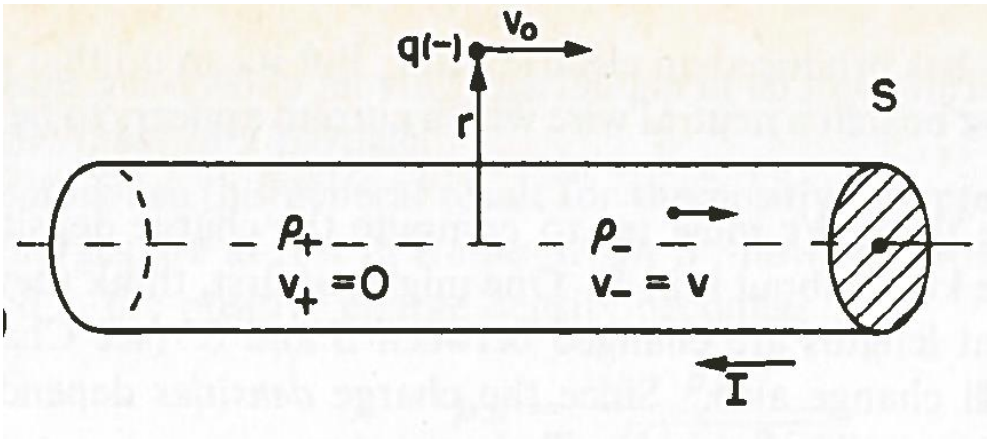

Figure 11: Charge $q(-)$ and current-carrying wire in laboratory frame **S** in which the wire is at rest.

a) (1 pt) What is the electric force F_{el} acting on charge $q(-)$ in frame **S**?

b) (1 pt) What is the magnetic force F_{mag} acting on charge $q(-)$ in frame **S**?

c) (1 pt) Draw a figure similar to Figure 11 for frame **S'** in which the charge $q(-)$ is at rest.

d) (1 pt) What is the magnetic force F'_{mag} acting on charge $q(-)$ in frame **S'**?

e) (1 pt) Derive that the positive charge density ρ'_+ in frame **S'** is given by

$$\rho'_+ = \frac{\rho_+}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

f) (1 pt) Similarly, derive that the negative charge density ρ'_- in frame **S'** is given by

$$\rho'_- = \rho_- \sqrt{1 - \frac{v^2}{c^2}}.$$

g) (1 pt) Give an expression for the total charge density ρ' in frame **S'** in terms of the positive charge density ρ_+ in frame **S**.

h) (1 pt) What is the electric force $F'_{\mathbf{el}}$ acting on charge $q(-)$ in frame **S'**?

i) (1 pt) Show that

$$F'_{\mathrm{el}} = \frac{F_{\mathrm{mag}}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

j) (1 pt) Explain that we obtain the same physical result in both frames.

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Thank you for your participation!

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