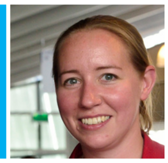
PION 2020

Natuurkunde Olympiade
13 november

Solutions

PION2020.nsaweb.nl


Nederlandse
Natuurkundige
Vereniging

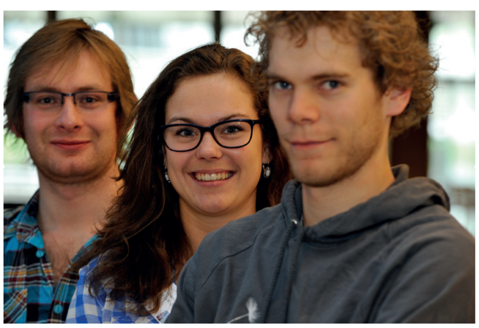
Tweedejaarsstudenten kunnen een jaar gratis lid worden. Alle andere studenten krijgen een flinke korting op de contributie.

Voordelen van het NNV-lidmaatschap:

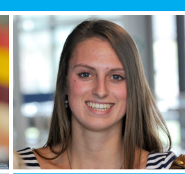
- Maandelijks het Nederlands Tijdschrift voor Natuurkunde in de bus
- Zeer aantrekkelijke ledenkorting op het jaarlijkse evenement FYSICA
- Optie op gratis lidmaatschap van de NNV-secties
- Toegang tot het complete digitale NTvN-archief
- Subsidie voor studiereizen en symposia van studieverenigingen
- Geassocieerd lidmaatschap van de European Physical Society
- Verbondenheid met de fysische gemeenschap!

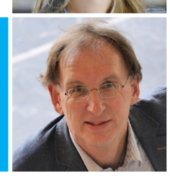
De Nederlandse Natuurkundige Vereniging bestaat al sinds 1921 en is dé vereniging voor natuurkundigen in Nederland. De NNV is voor alle fysici: studenten, fysici werkzaam in het bedrijfsleven, onderwijs, academia... Immers: Eenmaal fysicus, altijd fysicus!

Postbus 41882, 1009 DB Amsterdam
T: 020-5922211
E: bureau@nnv.nl

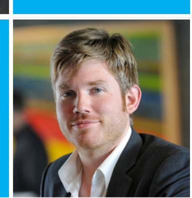



Gezicht van de natuurkunde




www.nnv.nl

Contents

1 The power of accretion

Prof. dr. S.B. (Sera) Markoff
Anton Pannekoek Institute of Astronomy - University of Amsterdam
10 points

(a) Consider hydrogen gas captured at a very large distance, in orbit around a maximally rotating 10 solar mass black hole. Assume it eventually reaches the event horizon with radius $R_{\rm hor} = GM_{\rm BH}/c^2$, where $M_{\rm BH}$ is the mass, and that the energy gained on infall is converted into internal heat. What is the average temperature of the gas?
Solution: The potential energy of the gas would be,

$$E = -\frac{GMm}{R} \tag{1}$$

Now, the energy loss is $\Delta E = E(R) - E(\infty)$. We get,

$$\Delta E = -\frac{GMm}{R} \tag{2}$$

Boltzmann constant $k_{\rm B} = 1.38 \times 10^{-16}$ erg K^{-1}, solar mass $M_\odot = 2 \times 10^{33}$ g, electron mass $m_{\rm e} = 9.1 \times 10^{-28}$ g, proton mass $m_{\rm p} = 1.67 \times 10^{-24}$ g, gravitational constant $G = 6.67 \times 10^{-8}$ cm^3 g^{-1} s^{-2} and the event horizon radius for a maximally rotating black hole is $R_{\rm hor} = GM/c^2$, where $c = 3 \times 10^{10}$ cm s^{-1} is the speed of light.

$$\Delta E = \frac{GM(m_{\rm p} + m_{\rm e})}{R_{\rm hor}} \approx \frac{GMm_{\rm p}}{R_{\rm hor}} \tag{3}$$

Use $R_{\rm hor} = GM/c^2$, we get $\Delta E = m_{\rm p}c^2$. Equating this to thermal energy $\Delta E_{\rm th} = 2 \times (3/2)k_{\rm B}T$ (a factor "2" for the two particles- one electron and one proton). Thus we get,

$$T = \frac{m_{\rm p}c^2}{3k_{\rm B}} \approx 10^{12}\text{K} \tag{4}$$

Order-of-magnitude estimates are okay.

(b) If the gas is fully ionised it will form a plasma. Consider plasma near the black hole with velocity $\vec{V}$, electric field $\vec{E}$ and magnetic field $\vec{B}$ (all quantities are measured in the lab frame). In ideal magnetohydrodynamics (MHD), the plasma is assumed to be highly conducting (conductivity $\sigma \to \infty$). Prove that the Lorentz force on a charged particle vanishes in the comoving frame. Using the result of your proof, find the relationship between $\vec{E}$, $\vec{V}$, and $\vec{B}$ in the lab frame. [*Hint: What is the force in the lab frame?*]

Solution:
Using Ohm's law $\vec{J}' = \sigma\vec{E}'$, we can see that even a small electric field $\vec{E}'$ in the comoving frame can give rise to large currents, which will initialize charge flow and rapidly lead to neutrality. Thus the electric field shorts itself out. By definition, in the comoving frame the plasma is at rest, so the magnetic force also vanishes. As a result, for an arbitrary charge q the Lorentz force law yields

$$\vec{F}' = q\vec{E}' + q\frac{\vec{V}'}{c} \times \vec{B}' = 0. \tag{5}$$

From eq. 5, as there is no acceleration in the comoving frame, there will be no acceleration observed in any inertial frame. Thus the Lorentz force law in the lab frame yields

$$\vec{F} = q\vec{E} + q\frac{\vec{V}}{c} \times \vec{B} = 0 \Rightarrow \vec{E} = -\frac{\vec{V}}{c} \times \vec{B} \tag{6}$$

(c) Around black holes, magnetic fields can actually dynamically dominate the flow. Show that under the additional assumptions of axisymmetry ($\partial/\partial\phi \to 0$) and time–independence ($\partial/\partial t \to 0$), $\vec{E}_\phi$ vanishes. In MHD it is convenient to decompose vectors in the form: $\vec{B} = B_p\hat{p} + B_\phi\hat{\phi}$ where the B_p is called the poloidal component of the magnetic field (in the $R - z$ plane) and B_ϕ is the toroidal component. Now, using the decomposed vectors and $E_\phi = 0$, prove that $\vec{V}_p = \kappa \vec{B}_p$ where κ is a proportionality constant.

Solution:
The electric field can be expressed in terms of the electromagnetic potentials, which under the assumption of time independence gives us

$$\vec{E} = -\nabla\Phi - \frac{\partial A}{c\partial t} = -\nabla\Phi, \tag{7}$$

where Φ is the scalar potential. The assumption of axisymmetry gives

$$E_\phi = -\frac{1}{R}\frac{\partial\Phi}{\partial\phi} = 0., \tag{8}$$

where R is the cylindrical radius.

On expressing $\vec{V} = \vec{V}_p + \vec{V}_\phi$ and $\vec{B} = \vec{B}_p + \vec{B}_\phi$, in terms of their poloidal and toroidal components (denoted by the subscripts p and ϕ respectively), the cross product now yields

$$-c\vec{E} = \vec{V} \times \vec{B} = \vec{V}_p \times \vec{B}_p + \vec{V}_p \times \vec{B}_\phi + \vec{V}_\phi \times \vec{B}_p + \vec{V}_\phi \times \vec{B}_\phi \tag{9}$$

Only the first terms' cross product yields a ϕ directed vector. Thus, we have $-c\vec{E}_\phi = \vec{V}_p \times \vec{B}_p = 0$. For the cross product of two non–zero vectors to vanish, they must be parallel and therefore

$$\vec{V}_p = \kappa \vec{B}_p, \tag{10}$$

where κ is some constant. To summarize: the poloidal components of the magnetic field and velocity are aligned!

Black holes systems (and other compact objects) are known to launch magnetised jets of plasma along their poles. Why?

(d) Using conservation laws, we can show that $V_\phi - \kappa B_\phi = R\Omega$ (where κ is the same proportionality constant for part (c) and Ω is the local angular speed) and use it for a simplified expression of the electric field. Now compute the Poynting flux from the plasma using the expressions for the electric and magnetic fields you obtained in the previous problem.

Solution:

The Poynting flux is given by:

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B}. \tag{11}$$

Using eq. 9 the non zero electric field terms are

$$-c\vec{E} = \vec{V}_p \times \vec{B}_\phi + \vec{V}_\phi \times \vec{B}_p. \tag{12}$$

Let us define the unit vector along the poloidal direction to be $\hat{p}$ and along the toroidal direction as $\hat{\phi}$. The unit vector normal to the plane containing these two unit vectors is defined as

$$\hat{n} = \hat{p} \times \hat{\phi}. \tag{13}$$

Thus the electric field can be written as

$$-c\vec{E} = \kappa B_p B_\phi \hat{n} - V_\phi B_p \hat{n} = B_p(\kappa B_\phi - V_\phi)\hat{n}. \tag{14}$$

Working the $\vec{E} \times \vec{B}$ term using $\vec{E} = \frac{B_p \Omega R}{c}\hat{n}$, we get

$$\vec{E} \times \vec{B} = \vec{E} \times \vec{B}_p + \vec{E} \times \vec{B}_\phi = -\frac{B_p B_\phi \Omega R}{c}\hat{p} + \frac{B_p^2 \Omega R}{c}\hat{\phi} \tag{15}$$

(e) Calculate the electromagnetic power transported by the Poynting flux in the poloidal direction passing through an area of arbitrary radius R. Use the fact that angular momentum conservation provides a relation between the poloidal and toroidal components of the magnetic field: $B_\phi = \Omega R B_p/c$.

Solution:
Power due to Poynting flux: $P_{\rm EM} = |\vec{S}.\Delta\vec{A}|$, where $\Delta\vec{A} = \pi R^2 \hat{p}$

$$P_{\rm EM} = \frac{c}{4\pi}\frac{B_p B_\phi \Omega R}{c}\pi R^2 \tag{16}$$

Using $B_\phi = \Omega R B_p/c$, we have

$$P_{EM} = (\Omega R^2 B_p)^2/4c \tag{17}$$

(f) The upper limit of this electromagnetic power will be when all of the gravitational potential energy released goes into the magnetic energy. For this limiting value, calculate the strength of the poloidal magnetic field (B_p) rotating with the Keplerian angular velocity ($\Omega_{\rm K} = \sqrt{GM_{\rm BH}/R^3}$) over the event horizon area of a maximally rotating 10 solar mass black hole. Assume a mass accretion rate $\dot{M} = 2.2 \times 10^{-8}$ solar masses per year (i.e., just a tiny fraction of captured mass from a companion star).

Solution: The Keplerian angular velocity is $\Omega_{\rm K} = \sqrt{GM/R^3}$. Thus we have,

$$P_{\rm EM} = (\Omega R^2 B_p)^2/4c = \frac{1}{4c}\frac{GM}{R^3}R^4 B_p^2 = \frac{GMRB_p^2}{4c} \tag{18}$$

Available gravitational power= $P_{\rm grav} = GM\dot{m}/(R_{\rm hor})$. Place $P_{\rm EM} = P_{\rm grav}$. maximally rotating black hole, and therefore, $R_{\rm hor} = GM/c^2$. For a 10 solar mass black hole with mass accretion rate $\dot{m} = 2.2 \times 10^{-8}$ solar masses per year $= 1.4 \times 10^{18}$ g s^{-1}, we get,

$$B_p^2 = \frac{4\dot{m}c}{2R_{\rm hor}^2} = \frac{2\dot{m}c^5}{G^2M^2} \Rightarrow B_p \approx 2 \times 10^8 \mathrm{G} \tag{19}$$

For comparison, the average magnetic field of our sun is of the order of 1-10 Gauss. So, you can see just how strong the magnetic fields are in the vicinity of black holes! Gravity ultimately powers electromagnetic fields that channel accretion power into, and accelerate, outflows and jets. This is why black holes and other compact objects like pulsars are able to launch powerful outflows!!

Figure 1: The Universe

2 Higgs mechanism in electrodynamics

Prof. E. Laenen
Institute for Theoretical Physics - University of Amsterdam
10 points

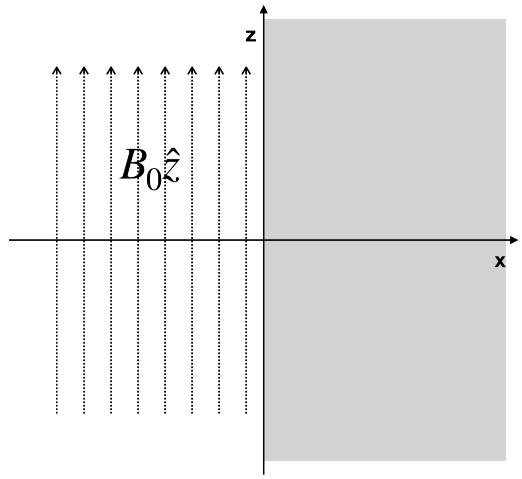


Figure 2

The Higgs mechanism of particle physics explains why certain particles have a mass rather than being massless like the photon propagating in vacuum. In this problem we shall see that such a situation already occurs in electrodynamics, for a magnetic field inside a BCS superconductor. In a BCS superconductor bound electron pairs (Cooper pairs) form a Bose condensate, which can act as a superconducting current.

Consider the setup of figure 2, in which there is a constant magnetic field pointing in the z direction outside the superconduting material. The material spans the entire region $x \geq 0$. A priori we know nothing about the magnetic field inside the material.

It is a special property of superconductors that the current density $\boldsymbol{j}$ inside is related to the vector potential in the material as

$$\boldsymbol{j} = -m_A^2 \boldsymbol{A}, \tag{20}$$

where m_A is a constant. In a more detailed analysis it turns out that $m_A = \mu_0 n_s e^2/m$ with n_s the number density of Cooper pairs, e, m the electron charge and mass, but that is of no relevance here. Note that for convenience we rescaled the current density $\boldsymbol{j} \equiv \mu_0 \boldsymbol{J}$.

a) (2 pts) Show that from (20) one can derive the second order differential equation

$$(\boldsymbol{\nabla}^2 - m_A^2)\boldsymbol{B} = 0\,. \tag{21}$$

Hint: You may wish to use the identity $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{C}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{C}) - \nabla^2 \boldsymbol{C}$.

Solution: Consider the curl of the current, and rewrite

$$\boldsymbol{\nabla} \times \mathbf{j} = -m_A^2 \boldsymbol{\nabla} \times \mathbf{A} = -m_A^2 \boldsymbol{B} \tag{22}$$

From the Maxwell equations we know $\boldsymbol{\nabla} \times \mathbf{B} = \boldsymbol{j}$. *Take the curl of this equation, use the hint, and again a Maxwell equation:* $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. *This leads to* (21).

b) (2 pts) Show that $B_x = B_y = 0$ inside the material.

*Solution: We use a boundary condition analysis. Imagine a gaussian volume that extends on both sides of the interface in figure 2. By the integral form of $\boldsymbol{\nabla} \cdot \boldsymbol{B} =$ one concludes that B_x is zero inside the material, since is also zero outside the material. To show that $B_y = 0$ using the integral form of Ampère's law, imagine a Ampere loop in the y-direction extending on both sides of the interface (with vanishing extent in the x-direction). With $B_y = 0$ in air, we have that $B_y = 0$ inside the material as well. Of the same is **not** true for the z direction.*

c) (1 pts) Show that the general solution to the result of (21) can be written as $B_z(x) = C_1 \mathrm{e}^{\frac{x}{d}} + C_2 \mathrm{e}^{-\frac{x}{d}}$. How does d depend on m_A? How does this solution change if we add $\boldsymbol{\nabla}\lambda$ to $\boldsymbol{A}$ in (20), in which λ is an arbitrary function?

Solution: With the result of part b), we have that (21) reads $\partial_x^2 B_z(x) = m_A^2 B_z(x)$, to which

$$B_z(x) = C_1 \mathrm{e}^{\frac{x}{d}} + C_2 \mathrm{e}^{-\frac{x}{d}} \tag{23}$$

is the general solution, with $m_A = 1/d$. We have written it in this way to make the exponents manifestly dimensionless. Changing

$$\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\lambda \tag{24}$$

does not change the solution, since $\boldsymbol{\nabla} \times \boldsymbol{A}$ does not change. This change of the vector potential without physical consequences is called a gauge transformation.

d) (1 pts) Why must C_1 vanish? Use boundary conditions to fix C_2.

Solution: Clearly at $x = +\infty$ we want the solution to be finite, hence $C_1 = 0$. Using that $B_z(0) = B_0$ (see figure) we have $C_2 = B_0$.

e) (2 pts) Check that your solution is consistent. I.e. given your result for the magnetic field, calculate the current density $\boldsymbol{j}$, use (20) to obtain the vector potential $\boldsymbol{A}$ and consequently calculate the magnetic field $\boldsymbol{B}$.

Solution: Since $\boldsymbol{j} = \boldsymbol{\nabla} \times \mathbf{B}$ we can compute this curl and find $m_A B_0 \mathrm{e}^{-x m_A} \hat{\boldsymbol{y}}$. Since also $\boldsymbol{j} = -m_A^2 \boldsymbol{A}$ we have

$$\boldsymbol{A} = -\frac{1}{m_A} B_0 \mathrm{e}^{-x m_A} \hat{\boldsymbol{y}} \,,$$

so that $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = B_0 \mathrm{e}^{-x m_A} \hat{\boldsymbol{z}}$ indeed.

f) (2 pts) Sketch the magnetic field for $x = -5d$ to $x = 5d$. Indicate the characteristic attenuation length $x = d$ in the plot.

Solution: the sketch can look roughly as follows

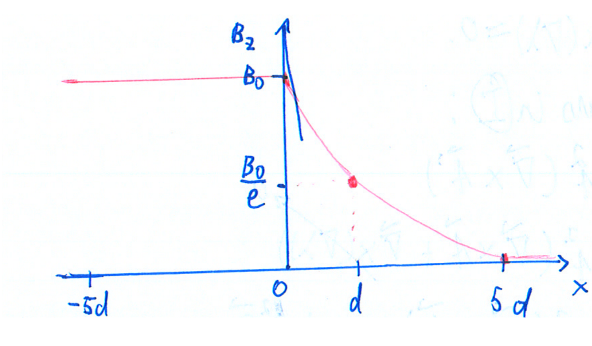


Figure 3

The result of our analysis is that magnetic fields are absent inside a superconductor, except in a very thin boundary layer. This phenomenon is known as the Meissner effect. The parameter m_A functions as an effective photon mass. Forces mediated by massive particles are in general short-range, hence the Meissner effect can be understood as the photon acquiring a mass inside the superconductor.

3 QUESTION M (Microtubule Length Control)

Prof. dr. ir. E.J.G. (Erwin) Peterman
LaserLab - Vrije Universiteit
10 points

a) These arguments can be made more systematically by appealing to simple rate equations of the form

$$n_i(t+\Delta t) = n_i(t) + k_{bind}\Delta t + n_{i-1}(t)v\Delta t - n_i v\Delta t, \tag{25}$$

which can be rewritten as

$$\frac{dn_i}{dt} = k_{bind} + n_{i-1}v - n_i v. \tag{26}$$

In steady state, this implies that

$$n_i = n_{i-1} + \frac{k_{bind}}{v}. \tag{27}$$

This difference equation has the solution already quoted above. A second way to the same result is to exploit the approximation

$$n_{i-1} \approx n_i - \frac{\partial n}{\partial x}a \tag{28}$$

Resulting in

$$\frac{dn}{dx} = \frac{k_{bind}}{av} \tag{29}$$

which implies that

$$n(x) = \frac{k_{bind}}{av}x = i\frac{k_{bind}}{v}, \tag{30}$$

precisely as was asserted earlier. (N.B. $k_{step} = v$)

b)

$$n_i = n_{i-1} + \frac{k_{bind}}{v} \tag{31}$$

Hence

$$n_1 = \frac{k_{bind}}{v},\ n_2 = \frac{k_{bind}}{v} + \frac{k_{bind}}{v},\ n_N = \frac{Nk_{bind}}{v}, \tag{32}$$

The rate out of the last monomer is $n_N \times k_{step}$, which is equal to $Nk_{bind} = \frac{k_{bind}L}{a}$, since $L = Na$, which is equal to the depolymerisation rate, which is proportional to microtubule length L. The microtubule shortening velocity is equal to $k_{bind}L$.

c)

$$\frac{dL(t)}{dt} = ak_{on} - k_{bind}L \tag{33}$$

In steady state:

$$\frac{dL(t)}{dt} = ak_{on} - k_{bind}L \rightarrow ak_{on} = k_{bind}L^* \rightarrow L^* = \frac{ak_{on}}{k_{bind}} = \frac{1\mu m/min}{0.05/min} = 20\mu m \tag{34}$$

4 Gravitoelectromagnetism and AMO physics

Prof. dr. H.L. Bethlem and W. van der Meer, M.Sc.
Quantum Metrology and Laser Applications – Vrije Universiteit Amsterdam
10 points

a) The difference in gravitational potential between a point at the surface of the earth and a point Δh higher is

$$\Delta\Phi_g = -GM\left(\frac{1}{R+\Delta h}-\frac{1}{R}\right) = g\Delta h.$$

The resulting fractional frequency shift, $\delta f/f = \Delta\Phi_g/c^2$. With $g = 9.81$ m/s^2 and $\Delta h = 0.33$ m, we find $\delta f/f = 3.6\times10^{-17}$.

b) We find the gravitomagnetic field of the earth by using the GEM version of the Biot-Savart law:

$$\vec{B}_g = \frac{G}{c^2}\int_{volume}\frac{\vec{J}_g\times\hat{r}}{r^2}dV.$$

Our problem is essentially the same as determining the magnetic field of a rotating homogeneously charged sphere. We divide up the earth in a series of concentric current loops. We first look at a ring of radius r, at a position z along the axis (i.e. we choose $z = 0$ at the center of the earth, so the distance from a ring at position z to the North Pole is $R - z$ with R being the radius of the earth). The ring has width dr and dz. The mass of this ring is

$$dm = \frac{M}{\frac{4}{3}\pi R^3}2\pi r dr dz = \frac{3M}{2R^3}r dr dz,$$

with M being the mass of the earth.
The mass current through such a ring (the mass that passes a certain point per second) is equal to

$$dI_g = \frac{dm}{dt} = \frac{3M}{2R^3}\frac{1}{T}r dr dz,$$

with T the rotation period of the earth (i.e., the number of seconds in a day).
The contribution of a ring to the gravitomagnetic field at the North Pole is equal to:

$$\begin{aligned} dB_{gz} &= \frac{G}{c^2}dI_g\frac{2\pi r}{(R-z)^2+r^2}\frac{r}{\sqrt{(R-z)^2+r^2}} \\ &= \frac{G}{c^2}\frac{3M}{2R^3T}r dr dz\frac{2\pi r}{(R-z)^2+r^2}\frac{r}{\sqrt{(R-z)^2+r^2}} \end{aligned}$$

The total gravitomagnetic field is found by integrating over z and r:

$$\begin{aligned} B_g &= \frac{3\pi GM}{R^3Tc^2}\int_{-R}^{R}\int_0^{\sqrt{R^2-z^2}}\frac{r^3}{((R-z)^2+r^2)^{3/2}}dr dz \\ &= \frac{3\pi GM}{R^3Tc^2}\frac{4}{15}R^2 \\ &= \frac{4\pi GM}{5RTc^2} \end{aligned}$$

Using $G = 6.67428\times10^{-11}$ Nm^2/kg^2, $M = 5.972\times10^{24}$ kg, $R = 6371\times10^3$ m and $T = 24*60*60 = 86400$ s, we find $B_g = 2.025\times10^{-14}$ Hz.
In order to avoid all tedious integrals, one could assume that all mass is concentrated in an infinetely thin ring at the equator with a radius of (say) $\frac{1}{2}R$. The mass current through this ring is simply

$$I_g = \frac{M}{T}.$$

The gravitomagnetic field at the North Pole due to this ring is equal to (with $z = 0$ and $r = \frac{1}{2}R$):

$$
\begin{aligned}
B_{gz} &= \frac{G}{c^2} I_g \frac{2\pi r}{(R-z)^2 + r^2} \frac{r}{\sqrt{(R-z)^2 + r^2}} \\
&= \frac{G}{c^2} \frac{M}{T} \frac{\pi R}{R^2 + \frac{1}{4}R^2} \frac{\frac{1}{2}R}{\sqrt{R^2 + \frac{1}{4}R^2}} \\
&= \frac{G}{c^2} \frac{M}{T} \frac{\pi}{\frac{5}{4}R} \frac{\frac{1}{2}}{\sqrt{\frac{5}{4}}} \\
&= \frac{8\pi G M}{5\sqrt{5} R T c^2}
\end{aligned}
$$

So we find a $2/\sqrt{5} \sim 0.89$ times smaller value than found by solving the correct integral.

c) We find the gravitomagnetic dipole moment of a hydrogen molecule by realising that

$$
\mu_g = I_g A = \frac{2m_P}{T_{H_2}} \pi r_0^2.
$$

With the equilibrium radius of hydrogen taken to be $r_0 = 0.1$ nm (it is actually slightly smaller), the rotational period, $T_{H_2} = 1/f_{H_2} = 3 \times 10^{-12}$ s, and the mass of the proton, $m_P = 1.672 \times 10^{-27}$ kg, we find $\mu_g = 3.15 \times 10^{-34}$ m^{2}kg/s.

d) The energy difference between clockwise and anti-clockwise spinning molecules (rotating in opposite direction or in the same direction as the earth, respectively), is $W_g = 2\mu_g B_g = 1.27 \times 10^{-47}$ J, corresponding to a frequency difference, $\delta f = 1.92 \times 10^{-14}$ Hz, or a fractional difference, $\delta f/f = 6.4 \times 10^{-27}$, unlikely to be ever measured.

5 Nature's Near-Perfect Clocks

Dr. J.W.T. (Jason) Hessels
High Energy Astrophysics Radio Astronomy - UvA ASTRON
10 points

Solution: Pulsar timing b)

We need to consider by how much the model-predicted versus observed rotational phase, ϕ, have deviated in time:

$\Delta\phi = \frac{1}{2}\dot{\nu}\Delta t^2$.
The accumulated residual delay is 6000ms. Dividing by the rotational period, $P = 650$ ms, we get the number of rotational cycles missed in the modeled $\dot{\nu}$: $\Delta\phi = 9.2$. The time span $\Delta t = 4000\text{days} = 345600000\text{s}$.
Therefore, the model $\dot{\nu}$ is in error by $\dot{\nu}_{\text{err}} = 1.5 \times 10^{-16}\ s^{-2}$.

That is equivalent to $\dot{P}_{\text{err}} = 6.5 \times 10^{-17}$ s/s.
The modeled $\dot{P}$ was thus off by only $\frac{6.5\times10^{-17}}{4\times10^{-14}} = 0.2\%$. This demonstrates that timing of this pulsar must be capable of determining the spin-down rate to at least a part in a thousand.
Since the residuals are becoming increasingly position with time that means that we were underestimating the $\dot{P}$ and that the true value is 4.0065×10^{-14} s/s.

Solution: Pulsar timing c)

We see that the residuals have a sinusoidal shape with a period of 365 days. This tells us that there is an error in the position being used in the pulsar timing model. The amplitude of the sinusoid is ~ 70 ms. If the pulsar's position is perfectly known, then it should be possible to perfectly barycenter the TOAs (i.e. reference the arrival times to the position of the solar system barycenter). (NB: the geometry is simplified because we've assumed for this calculation that this pulsar happens to be in the Ecliptic) The residuals shown here tell us that the position is off such that the barycentric correction is being made for a position that corresponds to a point in the Earth's orbit around the Sun in which it is ~ 70 ms early/late compared to where it should be (see Figure 4). $\Theta_{\text{rad}} = s/r = 70 \times 10^{-3}\ \text{s} \cdot 300000\ \text{km s}^{-1}/150 \times 10^6\ \text{km} = 0.00014$ rad. This is equivalent to 29 arcseconds. If the individual TOA precision is 0.1 ms, then we should be sensitive to sinusoids in the residuals (due to an incorrect position) with an amplitude of ~ 0.1 ms (on that order). This corresponds to a positional precision of ~ 29 arcsecond/(70 ms/0.1 ms) = 0.04 arcsecond. NB: even higher-precision positions are now routinely derived using timing of millisecond pulsars.

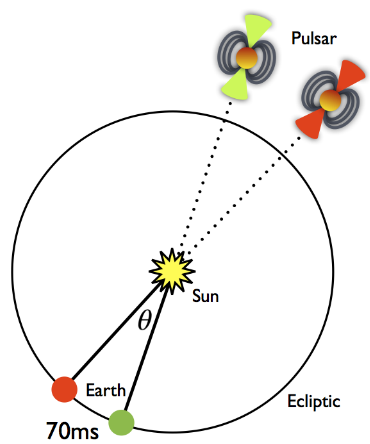


Figure 4: Schematic showing how to consider the positional offset. The green position of the pulsar is correct, whereas the red position of the pulsar is what is being used in the pulsar ephemeris.

6 The Kapitza Pendulum

Prof. dr. J.S. (Jean-S ebastien) Caux
Institute for Theoratical Physics – University of Amsterdam
10 points

a) [2 pts: 1 for kinetic energy, 1 for potential energy]

Let $x(t), y(t)$ be the coordinates of the pendulum's mass. In terms of the angle $\phi(t)$, we have

$$x(t) = l \sin \phi(t), \qquad y(t) = d \sin \omega t - l \cos \phi(t).$$

Using the dot notation for time derivatives,

$$\dot{x}(t) = l\dot{\phi}(t) \cos \phi(t), \qquad \dot{y}(t) = \omega d \cos \omega t + l\dot{\phi}(t) \sin \phi(t).$$

The kinetic energy is given by

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{ml^2}{2}\dot{\phi}^2 + m\omega ld\ \dot{\phi} \sin \phi \cos \omega t + \frac{m}{2}\omega^2 d^2 \cos^2 \omega t.$$

For the potential energy, we have (up to an inconsequential constant)

$$V = mgy = mgd \sin \omega t - mgl \cos \phi.$$

b) [1 pt]

Using the Lagrangian $L = T - V$, we obtain

$$\frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi} + m\omega ld \sin \phi \cos \omega t \to \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = ml^2\ddot{\phi} + m\omega ld\dot{\phi} \cos \phi \cos \omega t - m\omega^2 ld \sin \phi \sin \omega t,$$
$$\frac{\partial L}{\partial \phi} = m\omega ld\dot{\phi} \cos \phi \cos \omega t - mgl \sin \phi.$$

Equating these expressions gives the expected equation of motion.

c) [1 pt]

By imagining the effect of rapidly moving the suspension point, it becomes clear that the correct expression is

$$2)\ \phi_f(t) = -\frac{d}{l} \sin \phi_s \sin \omega t.$$

Namely, if the angle ϕ is zero, the angle does not change (excluding option 1). As the suspension moves up/down, the angle's absolute value is reduced/increased (excluding option 3).

d) [2 pts: 1 for $\sin \phi$ expression, 1 for final expression]

For $d \ll l$, we can write (using the expression for ϕ_f from the previous sub-question, remembering the fact that $\phi_f \propto \frac{d}{l}$ so is small)

$$\sin \phi = \sin(\phi_s + \phi_f) = \sin \phi_s \cos \phi_f + \cos \phi_s \sin \phi_f \approx \sin \phi_s \left(1 - \frac{d}{l} \cos \phi_s \sin \omega t\right).$$

Direct substitution in the equation of motion gives the required answer.

e) [2 pts: 1 for evaluation of time averages, 1 for final expression]

To average over fast oscillations when ω is large, we consider averaging each term in the equation of motion over one fast time period:

$$\frac{\omega}{2\pi}\int_t^{t+\frac{2\pi}{\omega}} dt'(...).$$

We have

$$\frac{\omega}{2\pi}\int_t^{t+\frac{2\pi}{\omega}} dt' \sin\omega t' = 0, \qquad \frac{\omega}{2\pi}\int_t^{t+\frac{2\pi}{\omega}} dt' \cos\omega t' = 0,$$
$$\frac{\omega}{2\pi}\int_t^{t+\frac{2\pi}{\omega}} dt' = 1, \qquad \frac{\omega}{2\pi}\int_t^{t+\frac{2\pi}{\omega}} dt' \sin^2\omega t' = \frac{\omega}{2\pi}\int_t^{t+\frac{2\pi}{\omega}} dt' \frac{1}{2}(1-\cos 2\omega t') = \frac{1}{2}.$$

Applying these to the equation of motion (assuming that the slow degrees of freedom are constant over one fast period) means that only 3 terms survive, namely the ones given in the question.

f) [2 pts: 1 for giving the equilibria, 1 for their stability]

The equation of motion for the slow degrees of freedom can be written as

$$\frac{d}{dt}\frac{\partial L_{\text{eff}}}{\partial\dot\phi_s} = \frac{\partial L_{\text{eff}}}{\partial\phi_s}, \qquad L_{\text{eff}} = T - V_{\text{eff}}$$

with the given effective potential, as can easily be verified by direct substitution.
The minima of the effective potential are determined from the condition for the equilibrium angle ϕ_s^e

$$\left.\frac{\partial V_{\text{eff}}}{\partial\phi_s}\right|_{\phi_s^e} = 0 \quad\longrightarrow\quad \sin\phi_s^e\left(\frac{g}{l} + \left(\frac{\omega d}{2l}\right)^2 2\cos\phi_s^e\right) = 0$$

which means that either

$$a)\ \sin\phi_s^e = 0 \qquad \text{or} \qquad b)\ \cos\phi_s^e = -\frac{2gl}{\omega^2 d^2}.$$

If the frequency ω is below $\omega_c \equiv \sqrt{2gl}/d$, only the a (left) condition can be fulfilled (the trigonometric functions must have absolute value ≤ 1). We then find two minima: $a_1 : \phi_s = 0$ and $a_2 : \phi_s = \pi$. Since

$$\frac{\partial^2 V_{\text{eff}}}{\partial\phi_s^2} = mgl\cos\phi_s + \frac{m}{2}\omega^2 d^2\cos 2\phi_s$$

we have

$$\left.\frac{\partial^2 V_{\text{eff}}}{\partial\phi_s^2}\right|_{a_1} = mgl + \frac{m}{2}\omega^2 d^2, \qquad \left.\frac{\partial^2 V_{\text{eff}}}{\partial\phi_s^2}\right|_{a_2} = -mgl + \frac{m}{2}\omega^2 d^2.$$

For $\omega < \omega_c$, the left expression is positive, but the right one is negative. We thus conclude that the pendulum's equilibrium position is thus either $a_1 : \phi_s = 0$ (stable) or $a_2 : \phi_s = \pi$ (unstable). If the frequency ω goes above the critical value ω_c however, we immediately see that solution a_2 becomes stable (solution a_1 always remains stable).

Furthermore, for $\omega > \omega_c$, the b condition above can also be fulfilled, the equilibrium angle then taking the value

$$\cos\phi_s^e = -\frac{\omega_c^2}{\omega^2} \quad\rightarrow\quad b_1 : \phi_s^e = \operatorname{acos}\left(-\frac{\omega_c^2}{\omega^2}\right), \qquad b_2 : \phi_s^e = \pi - \operatorname{acos}\left(-\frac{\omega_c^2}{\omega^2}\right).$$

Looking at the stability condition (using $\cos 2\phi = 2\cos^2\phi - 1$),

$$\left.\frac{\partial^2 V_{\text{eff}}}{\partial\phi_s^2}\right|_{b_{1,2}} = mgl\left(-\frac{\omega_c^2}{\omega^2}\right) + \frac{m}{2}\omega^2 d^2\left(2\frac{\omega_c^4}{\omega^4} - 1\right) = \frac{md^2}{2\omega^2}\left(\omega_c^4 - \omega^4\right)$$

shows that minima $b_{1,2}$ (which exist only for $\omega > \omega_c$) are always unstable.

Therefore, in summary, we have the stable equilibrium points

- $a_1 : \phi_s = 0$ for any ω
- $a_2 : \phi_s = \pi$ for $\omega > \omega_c = \sqrt{2gl}/d = \sqrt{2}\omega_0\frac{l}{d}$ (where $\omega_0 \equiv \sqrt{g/l}$ is the natural frequency of the pendulum).

7 Topological Mechanical Metamaterials

Dr. C.J.M. (Corentin) Coulais
Van der Waals-Zeeman Instituut - University of Amsterdam
10 points

a. Let us focus on one unit cell, located at site n, for which we assume without loss of generality that the rotor is pointing downwards (Fig. 5)ab. The geometry of the unit cell is described in Fig. 5. Importantly, the various geometric parameters are related by the following relations

$$\left\{ \begin{array}{lcl} \ell \cos\psi & = & p \\ \ell \sin\psi & = & 2r\cos\theta \end{array} \right. . \tag{35}$$

The torque induced by the tension of the $n-1^{\text{th}}$ spring is $\vec{\tau}_{n-1,n} = \vec{r}\vec{t}_{n-1}$, where $\vec{r} = r(\sin\theta\vec{e}_x - \cos\theta\vec{e}_y)$, is the

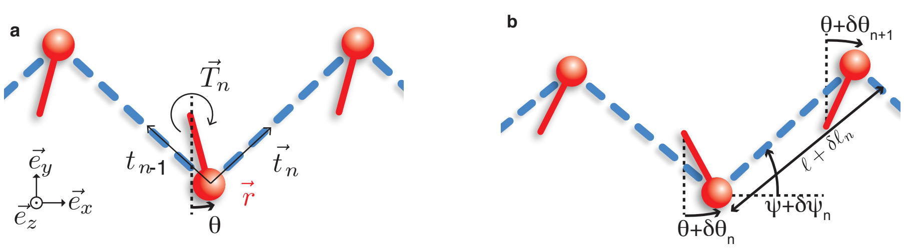


Figure 5: Sketches. (a) torque balance on unit cell n). (b) Geometry of the system in the deformed configuration.

vector describing the rotor and where the tension in the spring is $\vec{t}_{n-1} = -k\delta\ell_{n-1}(\cos\psi\vec{e}_x - \sin\psi\vec{e}_y)$, where $\delta\ell_{n-1}$ is the elongation of spring $n-1$. and the torque induced by the tension of the n^{th} spring is $\vec{\tau}_{n,n} = \vec{r}\vec{t}_n$, where the tension in the bond is $\vec{t}_n = k\delta\ell_n(\cos\psi\vec{e}_x + \sin\psi\vec{e}_y)$, where $\delta\ell_n$ is the elongation of spring n. Therefore, after a few algebraic manipulations, and using Eqs. (39) the resulting torque on rotor n, $\vec{T}_n = \vec{\tau}_{n-1,n} + \vec{\tau}_{n,n}$ reads

$$\vec{T}_n = k(a\delta\ell_n - b\delta\ell_{n-1})\vec{e}_z, \tag{36}$$

where the coefficients a and b are

$$\left\{ \begin{array}{lcl} a & = & r\cos\theta\frac{p+2r\sin\theta}{\sqrt{p^2+4r^2\cos^2\theta}} \\ b & = & r\cos\theta\frac{p-2r\sin\theta}{\sqrt{p^2+4r^2\cos^2\theta}} \end{array} \right. . \tag{37}$$

We now need to establish the relation between the spring elongation $\delta\ell_n$ and the rotations of the springs $\delta\theta_n$. To this end, we express the geometrical constraints in the deformed state, following Fig. 5

$$\left\{ \begin{array}{lcl} (\ell+\delta\ell_n)\cos(\psi+\delta\psi_n) & = & p - r(\sin(\theta+\delta\theta_n) - \sin(\theta+\delta\theta_{n+1})) \\ (\ell+\delta\ell_n)\sin(\psi+\delta\psi_n) & = & r(\cos(\theta+\delta\theta_n) + \cos(\theta+\delta\theta_{n+1})) \end{array} \right. , \tag{38}$$

which we then linearise to first order in ℓ_n, ψ_n and θ_n

$$\left\{ \begin{array}{lcl} (\ell+\delta\ell_n)\cos\psi - \ell\sin\psi\delta\psi_n & = & p - r\cos\theta(\delta\theta_n - \delta\theta_{n+1}) \\ (\ell+\delta\ell_n)\sin\psi + \ell\cos\psi\delta\psi_n & = & 2r\cos\theta - r\sin\theta(\delta\theta_n + \delta\theta_{n+1}) \end{array} \right. , \tag{39}$$

After a few algebraic manipulations, we find

$$\delta\ell_n = -a\delta\theta_n + b\delta\theta_{n+1}, \tag{40}$$

where a and b correspond to Eq. (37). Combining Eq. (36) and Eq. (40), we can now express the torque $\vec{T}_n$ as a function of the rotations of the springs as

$$\left\{ \begin{array}{lcl} \vec{T}_0 & = & k(-a^2\delta\theta_n + ab\delta\theta_{n+1})\vec{e}_z \\ \vec{T}_n & = & k(-(a^2+b^2)\delta\theta_n + ab(\delta\theta_{n-1} + \delta\theta_{n+1})\vec{e}_z \quad \text{for } n \in [2, N-1] \\ \vec{T}_N & = & k(-b^2\delta\theta_n + ab\delta\theta_{n-1})\vec{e}_z \end{array} \right. \tag{41}$$

Using Newton's law for angular momentum $Jd^2\delta\theta_n/dt^2 = \vec{T}_n \cdot \vec{e}_z$, where J is the second moment of inertia, we find Eqs. (**??**-**??**).

b. The solution to $-a\delta\theta_n + b\delta\theta_{n+1} = 0$ can be solved by recursion. $\delta\theta_2 = (a/b)\delta\theta_1$, $\delta\theta_3 = (a/b)^2\delta\theta_1$, $\delta\theta_n = (a/b)^{n-1}\delta\theta_1$.

c. The three angular displacement profiles are shown in Fig. 6. The zero mode is exponentially localized on the left (right) for $a < b$ ($a > b$), and is delocalised for $a = b$. Since the ratio $a/b = (p + 2r\sin\theta)/(p - 2r\sin\theta)$, the mode is localized on the left (right) for $\theta < 0$ ($\theta > 0$), *i.e.* the rotors are tilted to the left (right). The limit case where the mode is delocalised corresponds to $\theta = 0$, where the rotors are straight. The characteristic decay length of the zero mode diverges when the rotor's tilt goes to zero.

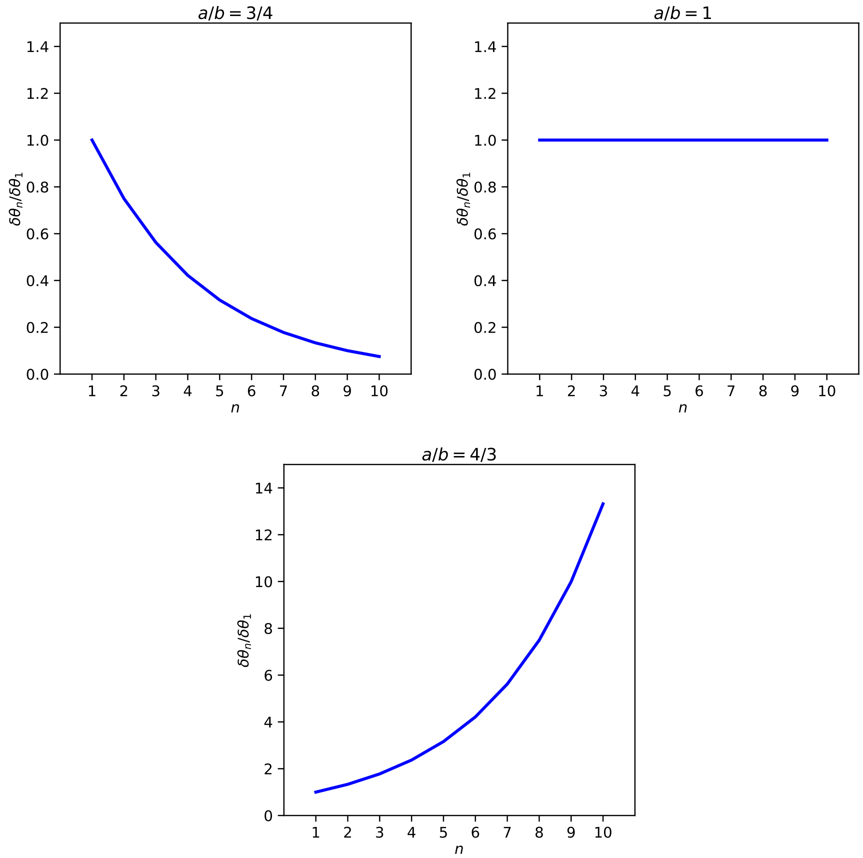


Figure 6: Displacement profiles of the zero-modes for $a = 0.75$, and $b = 1$ (left), $a = b = 1$ (middle) and $a = 1$ and $b = 0.75$ (right).

d. We inject the ansatz $\delta\theta_n = \delta\theta_0 \exp i(\omega t - qn)$ into Eq. (**??**) and find $-J\omega^2\delta\theta_0 = kr^2\cos^2\theta(-(a^2 + b^2) + ab(\exp(-iq) + \exp(iq))\delta\theta_0$. This leads to $\omega = \pm r\cos\theta\sqrt{\frac{k}{J}}\sqrt{D(q)}$, where

$$D(q) = (a^2 + b^2) - 2ab\cos q. \tag{42}$$

e. The three dispersion relations are shown in Fig. 7. For $a \neq b$—when the rotors are tilted—there is a gap centred at zero frequency. For $a = b$—when the rotors are straight—the gap disappears.

f. $D(q)$ can be rewritten as $D(q) = a^2 + b^2 - abe^{iq} - abe^{-iq}$ and factorized as $D(q) = R(q)Q(q)$ with

$$R(q) = a - be^{iq} \tag{43}$$

$$Q(q) = a - be^{-iq}. \tag{44}$$

g. The eigenvalues λ_1 and λ_2 of $H(q)$ have the following relations $\lambda_1 = -\lambda_2$ (because $H = 0$); and $\lambda_1\lambda_2 = -R(q)Q(q)$ (because $\det H = -R(q)Q(q)$). As a result, $\lambda_{1,2} = \pm\sqrt{R(q)Q(q)}$ are precisely equal to the frequencies $\omega(q)$ of the mechanical system. Thereby, the mechanical and the quantum systems are strictly analogue.

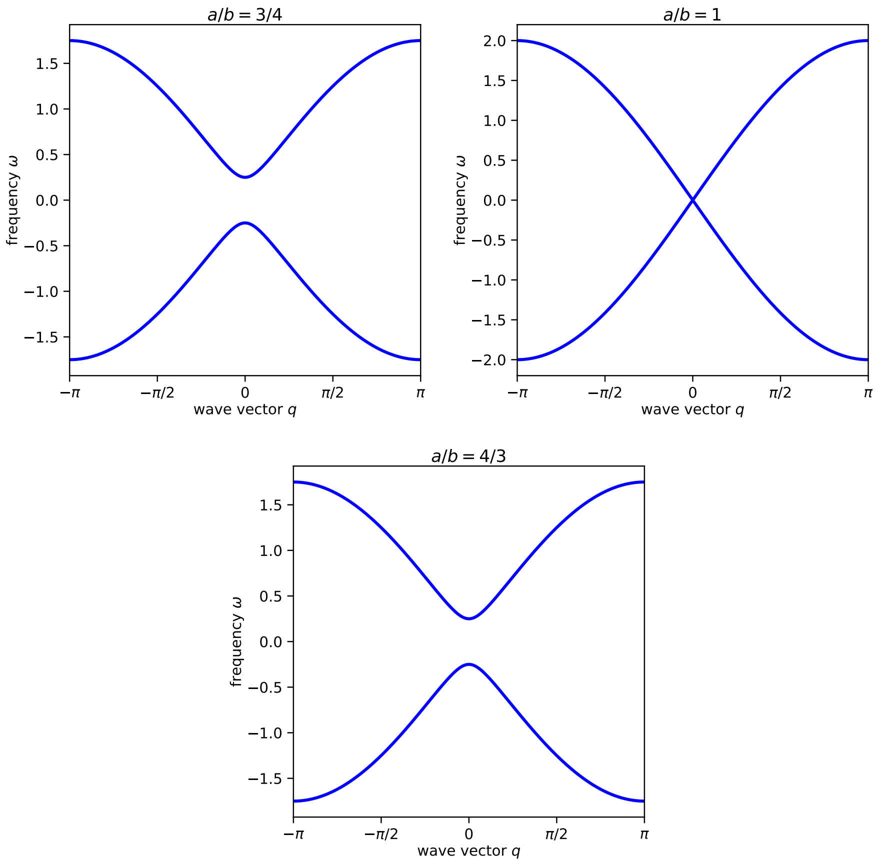


Figure 7: Dispersion relation ω vs. q for $a = 0.75$, and $b = 1$ (left), $a = b = 1$ (middle) and $a = 1$ and $b = 0.75$ (right).

h. The eigenvectors of $H(q)$ are given by $H(q)\psi_1(q) = \omega(q)\psi_1(q)$ and $H(q)\psi_2(q) = -\omega(q)\psi_2(q)$, hence are given by

$$\psi_1(q) = \begin{pmatrix} \sqrt{\frac{Q(q)}{R(q)}} \\ 1 \end{pmatrix} \tag{45}$$

$$\psi_2(q) = \begin{pmatrix} -\sqrt{\frac{Q(q)}{R(q)}} \\ 1 \end{pmatrix}. \tag{46}$$

$$\tag{47}$$

i. We first calculate $\partial\psi_{1,2}(q)/\partial q = (\pm\frac{1}{2}\sqrt{\frac{Q(q)}{R(q)}}\left(\frac{R'(q)}{R(q)} - \frac{Q'(q)}{Q(q)}\right), 0)^T$. The inner product becomes $\psi_{1,2}(q)\partial\psi_{1,2}(q)/\partial q = \frac{1}{2}\sqrt{\frac{Q^*(q)Q(q)}{R^*(q)R(q)}}\left(\frac{R'(q)}{R(q)} - \frac{Q'(q)}{Q(q)}\right)$ for both $\psi_1(q)$ and $\psi_2(q)$. As a result, the Berry connection becomes

$$\mathcal{A}(q) = \frac{i}{2}\left(\frac{bie^{iq}}{a - be^{iq}} + \frac{bie^{-iq}}{a - be^{-iq}}\right). \tag{48}$$

To calculate the topological invariant, we need to integrate $\mathcal{A}(q)$ as

$$\nu = -\frac{1}{4\pi}\int_0^{2\pi} dq \left(\frac{be^{iq}}{a - be^{iq}} + \frac{be^{-iq}}{a - be^{-iq}}\right). \tag{49}$$

Since the integrand is complex and has poles, we will use complex integrals. First we change variable, $q \to -q$ in the second integrand, and the integral becomes

$$\nu = -\frac{1}{2\pi}\int_0^{2\pi} dq \frac{be^{iq}}{a - be^{iq}}. \tag{50}$$

Second, we change variable $z = e^{iq}$, $dz/(iz) = dq$. The previous integral then becomes

$$\nu = -\frac{1}{2i\pi}\int_{\mathcal{C}} dz \frac{1}{\frac{a}{b} - z}, \tag{51}$$

where $\mathcal{C}$ is the unit circle in the complex plane. The integrand has one pole $z = a/b$. The Residues' theorem states that if this pole is outside of the integration contour, the integral is zero. This is the case for $a/b > 1$. In contrast if there is a pole comprised in the integration contour, the residue is finite and the integral is equal to $2i\pi(1/(\frac{a}{b} - z); z = \frac{a}{b}) = 2i\pi \lim_{z\to\frac{a}{b}}(z - \frac{a}{b})/(\frac{a}{b} - z) = -2i\pi$. As a result, we find

$$\nu = \begin{cases} 0 & \text{for } a > b \\ 1 & \text{for } a < b \end{cases} \tag{52}$$

The topological invariant is one (zero) when the mode is localized on the left (right) edge. The transition between the two corresponds to the limit case when the rotors are straight.

8 Spin dynamics due to mobile electron

Prof. dr. C.J.M. (Kareljan) Schoutens
Institute for Theoretical Physics – University of Amsterdam
10 points

a) For $t \ll g$, $t \ll h$, the leading terms in the probabilities we are looking for are of the form

$$P_{\Uparrow,\uparrow;\Downarrow}(t) \propto h^a g^b t^m, \quad P_{\Uparrow;\uparrow,\Downarrow}(t) \propto h^c g^d t^n \ .$$

Determine, without long calculations, the values a, b, m and c, d and n.

Starting from $\psi(0)$, to arrive at the state $\Uparrow,\uparrow;\Downarrow$ one needs the action of a term in H three times: hopping to the right (amplitude h), spin flip $\downarrow,\Uparrow \to \uparrow,\Downarrow$ at site R (amplitude $\propto g$) and hopping to the left (amplitude h). This translates to $a = 4$, $b = 2$ and hence (e.g. due to a dimensional argument) $m = 6$. For the other process, the hop to the left is left out and the leading term is $c = 2$, $d = 2$, $n = 4$.

b) Show that the total spin $\vec{S} = \vec{S_L} + \vec{s} + \vec{S_R}$ commutes with H. Organise the 16 states as eigenstates S^2 and S^z and give a list of the possible values of (S^2, S^z), including their respective multiplicity. Show that $\psi(0)$ belongs to a group of 4 eigenstates with equal values for S^2 en S^z. What are these values?

For both the L and R states, we have that

$$[1/2] \otimes [1/2] \otimes [1/2] = [3/2] \oplus [1/2] \oplus [1/2]$$

This gives us

$$\begin{aligned}
&(S^2, S^z) = (15/4, 3/2)_2, \quad (S^2, S^z) = (15/4, 1/2)_2,\\
&(S^2, S^z) = (15/4, -1/2)_2, \quad (S^2, S^z) = (15/4, -3/2)_2,\\
&(S^2, S^z) = (3/4, 1/2)_4, \quad (S^2, S^z) = (3/4, -1/2)_4.
\end{aligned}$$

The state $\psi(0)$ is singlet under $\vec{S_L} + \vec{s}$ and hence has $(S^2, S^z) = (3/4, 1/2)$.

c) Give a basis $\{e_1, e_2, e_3, e_4\}$ for the space found at b), with $e_1 = \psi(0)$ and e_3 the similar state for the electron at location R. Subsequently, write H as a 4×4 matrix with respect to this basis.

$$\begin{aligned}
e_1 &= \frac{1}{\sqrt{2}} \left(\Uparrow,\downarrow;\Uparrow - \Downarrow,\uparrow;\Uparrow \right),\\
e_2 &= \frac{1}{\sqrt{6}} \left(\Uparrow,\downarrow;\Uparrow + \Downarrow,\uparrow;\Uparrow \right) - \frac{2}{\sqrt{6}} \Uparrow,\uparrow;\Downarrow,\\
e_3 &= \frac{1}{\sqrt{2}} \left(\Uparrow;\downarrow,\Uparrow - \Uparrow;\uparrow,\Downarrow \right),\\
e_4 &= \frac{1}{\sqrt{6}} \left(\Uparrow;\downarrow,\Uparrow + \Uparrow;\uparrow,\Downarrow \right) - \frac{2}{\sqrt{6}} \Downarrow;\uparrow,\Uparrow
\end{aligned}$$

With

$$e_1 H_L e_1 = -\frac{3g}{4}, \qquad e_2 H_L e_2 = \frac{g}{4}, \quad \text{etc.}$$

and

$$\begin{aligned}
&e_1 H_K e_3 = h/2, \quad e_1 H_K e_4 = \sqrt{3}h/2,\\
&e_2 H_K e_3 = \sqrt{3}h/2, \quad e_2 H_K e_4 = -h/2,
\end{aligned}$$

one finds that

$$H = \begin{pmatrix} -\frac{3g}{4} & 0 & h/2 & \sqrt{3}h/2 \\ 0 & \frac{g}{4} & \sqrt{3}h/2 & -h/2 \\ h/2 & \sqrt{3}h/2 & -\frac{3g}{4} & 0 \\ \sqrt{3}h/2 & -h/2 & 0 & \frac{g}{4} \end{pmatrix}.$$

d) We now also use the symmetry P (parity), which exchanges L and R, and define

$$e_{1,\pm} = \frac{1}{\sqrt{2}}(e_1 \pm e_3), \qquad e_{2,\pm} = \frac{1}{\sqrt{2}}(e_2 \pm e_4)$$

so that

$$Pe_{i,\pm} = \pm e_{i,\pm}, \qquad i = 1, 2.$$

Show that in the basis $\{e_{1,+}, e_{2,+}, e_{1,-}, e_{2,-}\}$, H is a block matrix,

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix},$$

with

$$H_\pm = \begin{pmatrix} -\frac{3g}{4} \pm h/2 & \pm\sqrt{3}h/2 \\ \pm\sqrt{3}h/2 & \frac{g}{4} \mp h/2 \end{pmatrix}.$$

This follows from writing H in this basis.

e) Determine the evolution operators $U_H(t) = \exp(-iHt)$. Hint: first show that for a Hamiltonian M of the form

$$M = \begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

the evolution operator $U_M(t)$ is written as

$$U_M(t) = \exp[-iMt] = \cos(st)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\frac{\sin(st)}{s}\begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

with $s = \sqrt{a^2 + b^2}$.

First, shift $H_\pm$ as $H_\pm = \tilde{H}_\pm - g/4\,\mathrm{Id}$, which gives overall phase $e^{i(g/4)t}$. The matrices $\tilde{H}_\pm$ are of the form as given above with

$$\begin{aligned} a_+ &= -g/2 + h/2, \quad b_+ = \sqrt{3}h/2, \quad s_+ = \sqrt{(g/2 - h/2)^2 + 3h^2/2} \\ a_- &= -g/2 - h/2, \quad b_- = -\sqrt{3}h/2, \quad s_- = \sqrt{(g/2 + h/2)^2 + 3h^2/2} \end{aligned}$$

and give the evolution operators $U_\pm(t)$. The evolution operator for H then becomes

$$U_H(t) = e^{i(g/4)t}\begin{pmatrix} U_+(t) & 0 \\ 0 & U_-(t) \end{pmatrix}.$$

f) Now give $\psi(t)$ in the basis $\{e_{1,+}, e_{2,+}, e_{1,-}, e_{2,-}\}$.
As $\psi(0) = \frac{1}{\sqrt{2}}(e_{1,+} + e_{1,-})$, one finds

$$\begin{aligned} \psi(t) = &\frac{1}{\sqrt{2}}\left[(\cos(s_+t) - i\sin(s_+t)\frac{a_+}{s_+})e_{1,+} - i\sin(s_+t)\frac{b_+}{s_+})e_{2,+}\right] \\ &+\frac{1}{\sqrt{2}}\left[(\cos(s_-t) - i\sin(s_-t)\frac{a_-}{s_-})e_{1,-} - i\sin(s_-t)\frac{b_-}{s_-})e_{2,-}\right]. \end{aligned}$$

g) Determine the inner products $\langle\Uparrow,\uparrow;\Downarrow e_{i,\pm}$, $\langle\Uparrow;\uparrow,\Downarrow e_{i,\pm}$ and calculate $\langle\Uparrow,\uparrow;\Downarrow \psi(t)$ and $\langle\Uparrow;\uparrow,\Downarrow \psi(t)$.

$$\begin{aligned} &\Uparrow,\uparrow;\Downarrow e_{1,\pm} = 0, \quad \Uparrow,\uparrow;\Downarrow e_{2,\pm} = -\frac{1}{\sqrt{3}}, \\ &\Uparrow;\uparrow,\Downarrow e_{1,\pm} = \mp\frac{1}{2}, \quad \Uparrow;\uparrow,\Downarrow e_{2,\pm} = \pm\frac{1}{2\sqrt{3}} \end{aligned}$$

$$\begin{aligned}
{}_{\Uparrow,\uparrow;\Downarrow}\psi(t) &= \frac{i}{\sqrt{6}}\left[\sin(s_+t)\frac{b_+}{s_+} + \sin(s_-t)\frac{b_-}{s_-}\right] = \frac{ih}{2\sqrt{2}}\left[\frac{\sin(s_+t)}{s_+} - \frac{\sin(s_-t)}{s_-}\right] \\
{}_{\Uparrow;\uparrow,\Downarrow}\psi(t) &= -\frac{1}{2\sqrt{2}}\left[(\cos(s_+t) - i\sin(s_+t)\frac{a_+}{s_+}) - (\cos(s_-t) - i\sin(s_-t)\frac{a_-}{s_-})\right] \\
&\quad -\frac{i}{2\sqrt{6}}\left[\sin(s_+t)\frac{b_+}{s_+} - \sin(s_-t)\frac{b_-}{s_-}\right] \\
&= -\frac{1}{2\sqrt{2}}\left[\cos(s_+t) - \cos(s_-t)\right] - \frac{ig}{4\sqrt{2}}\left[\frac{\sin(s_+t)}{s_+} - \frac{\sin(s_-t)}{s_-}\right]
\end{aligned}$$

h) Now determine the leading terms in $P_{\Uparrow,\uparrow;\Downarrow}(t)$ and $P_{\Uparrow;\uparrow,\Downarrow}(t)$ for $t \ll g$, $t \ll h$ and compare the result with your prediction under a).

For small t

$$\begin{aligned}
{}_{\Uparrow,\uparrow;\Downarrow}\psi(t) &= \frac{ih}{2\sqrt{2}}\left[\frac{\sin(s_+t)}{s_+} - \frac{\sin(s_-t)}{s_-}\right] \\
&= \frac{ih}{2\sqrt{2}}\left[-\frac{1}{6}s_+^2t^3 + \frac{1}{6}s_-^2t^3)\right] + \mathcal{O}(t^5) \\
&= \frac{-ih}{12\sqrt{2}}\left[(g/2+h/2)^2 - (g/2-h/2)^2\right]t^3 = \frac{-ih^2g}{12\sqrt{2}}t^3 + \mathcal{O}(t^5), \\
{}_{\Uparrow;\uparrow,\Downarrow}\psi(t) &= -\frac{1}{2\sqrt{2}}\left[\cos(s_+t) - \cos(s_-t)\right] - \frac{ig}{4\sqrt{2}}\left[\frac{\sin(s_+t)}{s_+} - \frac{\sin(s_-t)}{s_-}\right] \\
&= \frac{1}{4\sqrt{2}}(s_+^2 - s_-^2)t^2 - \frac{ig}{4\sqrt{2}}\frac{-1}{6}(s_+^2 - s_-^2)t^3 + \mathcal{O}(t^4) \\
&= -\frac{hg}{4\sqrt{2}}t^2 - \frac{ihg^2}{24\sqrt{2}}t^3 + \mathcal{O}(t^4)
\end{aligned}$$

such that

$$P_{\Uparrow,\uparrow;\Downarrow}(t) = \frac{h^4g^2}{288}t^6 + \mathcal{O}(t^8), \quad P_{\Uparrow;\uparrow;\Downarrow}(t) = \frac{h^2g^2}{32}t^4 + \mathcal{O}(t^5)\ .$$

PION 2020 is made possible by:

- **PION 2020 committee**

Thomas Goldman	*Chair / Secretary*
Quinten Coelho	*Chair/ Secretary*
Siem Fenne	*Treasurer*
Christiaan van Asperen	*CEO (Chief Executive Opgaves)*
Adrian Ruesink	
Duncan Pelan	

- **Creators of the exercises**

1. Prof. dr. S.B. (Sera) Markoff
2. Prof. dr. E.L.M.P. (Eric) Laenen
   dr. A. (Anne) de Visser
3. Prof. dr. ir. E.J.G. (Erwin) Peterman
4. Prof. dr. H.L. Bethlem
   W. van der Meer, M.Sc.
5. Dr. J.W.T. (Jason) Hessels
6. Prof. dr. J.S. (Jean-Sébastien) Caux
7. Dr. C.J.M. (Corentin) Coulais
8. Prof. dr. C.J.M. (Kareljan) Schoutens

- **Reference committee:**

  - Prof. dr. Robbert Dijkgraaf
    Director and Leon Levy Professor at the Institute for Advanced Study in Princeton, New Jersey
  - Prof. dr. Gerard 't Hooft
    Winner of the 1999 Nobel Prize "for elucidating the quantum structure of electroweak interactions"
  - Prof. dr. Ralph Wijers
    Director of the Anton Pannekoek Instituut Amsterdam and Member of KNAW
  - Prof. dr. Vinod Subramaniam
    Rector Magnificus of the Vrije Universiteit Amsterdam
  - Prof. dr. Peter van Tienderen
    Dean of the Faculty of Science of the University of Amsterdam
  - Prof. dr. ir. Karen I.J. Maex
    Rector Magnificus of the University of Amsterdam

ASML

Nikhef

randstad

Delta
Institute for Theoretical Physics

Stichting Physica

Universiteit van Amsterdam

VU Vrije Universiteit Amsterdam

Faculty of Science
Physics and Astronomy

Utrecht University

TU/e Eindhoven University of Technology

Radboud University

Leiden Institute of Physics

VU Vereniging

university of groningen

faculty of science and engineering

UNIVERSITY OF TWENTE.

PION 2020

Natuurkunde Olympiade
13 november

Thank you for participation!

PION2020.nsaweb.nl