ANSWERS

PION | 2019

5E LUSTRUM

UNIVERSITY OF TWENTE.

nederlandse natuurkundige vereniging

TU/e APPLIED PHYSICS

UNIVERSITY OF AMSTERDAM
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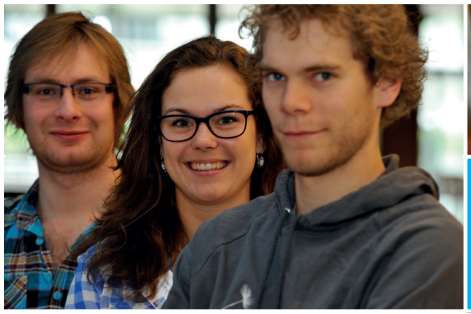

Gezicht van de natuurkunde

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Preface

Dear PION Participant,

PIONmeunig mooi that you're participating in this year's PION edition! We proudly present to you the exercises of PION2019. The committee has been busy the last couple of months to assemble this set of physics problems to challenge you. The problems cover a whole range of physics subjects, from fluid physics up to quantum computing. If your team is capable enough to solve these problems well enough, you can participate in this year's *Physics League Across Numerous Countries for Kick-ass Students* in Denmark. The committee wishes your team good luck, but most of all lots of fun!

From left to right: Frank Somhorst (problems), Aron van den Bogaard (secretary), Thies Jansen (chairman), Jardi Timmerhuis (treasurer), Lars Bossink (external affairs).

Rules

- **Please make every exercise on a separate sheet of paper stating your team's name! Please answer all questions in English.**
- It is not allowed to use any (study)books, notes, etc. other than BINAS and a dictionary.
- It is not allowed to communicate (about the problems) via any means with anyone other than team members.
- It is allowed to use a scientific, graphical calculator (Ti-83, Ti-84 or a similar model).
- Mobile phones are tolerated, but can only be used for emergencies, during the duration of the competition.
- The winner of PION2019 is the best team, with the most points. Three teams will get a ticket to PLANCKS: the best bachelor team and the two best teams that are not the best bachelor team.

In cases where these rules do not provide, the PION2019-committee will decide.

Contents

1 Graphene with a Twist

Prof. Dr. Ir. H.J.W. Zandvliet, Dr. K. Sotthewes, Dr. A. Van Houselt, Dr. E.S. Kooij
Physics of Interfaces and Nanomaterials - University of Twente
10 points

Graphene is a two-dimensional material consisting of sp^2 hybridized carbon atoms that are arranged in a honeycomb lattice (see Figure 2). The distance between nearest neighbour carbon atoms is a; the lattice constant amounts to $a\sqrt{3}$.

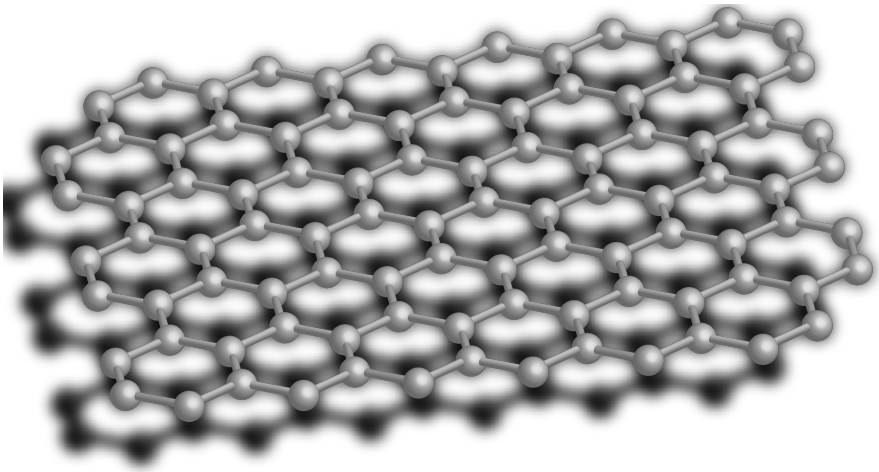
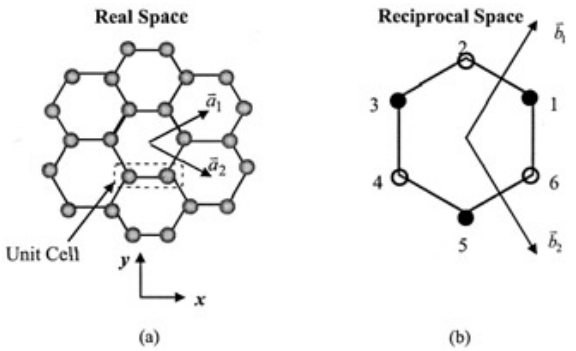


Figure 2: A single layer of graphene, in which carbon atoms are arranged in a honeycomb lattice.

The unit cell of graphene contains two carbon atoms. The reciprocal lattice can be constructed using the relation $\exp(i\mathbf{G}\cdot\mathbf{R}) = 1$, where $\mathbf{R}=\mathbf{a}_1+\mathbf{a}_2$ spans the lattice in real space and $\mathbf{G}=\mathbf{b}_1+\mathbf{b}_2$ the lattice in reciprocal space.

a) (2 pts) Determine the reciprocal lattice of graphene. Define lattice vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ of the real lattice and determine length and direction of the reciprocal vectors $\mathbf{b}_1$ and $\mathbf{b}_2$.

(2 pts) A schematic drawing of the real (a, left) and reciprocal (b, right) lattice of graphene:



Real lattice vectors are given by

$$\begin{aligned}\mathbf{a}_1 &= \frac{3a}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}a}{2}\hat{\mathbf{y}} \\ \mathbf{a}_2 &= \frac{3a}{2}\hat{\mathbf{x}} - \frac{\sqrt{3}a}{2}\hat{\mathbf{y}}\end{aligned}$$

To determine the reciprocal lattice we use the relation $\mathbf{b}_i \cdot \mathbf{a}_j = 2\pi\delta_{ij}$, from which we obtain:

$$\mathbf{b}_1 \cdot \mathbf{a}_2 = 0 \;\Rightarrow\; \mathbf{b}_{1,\hat{\mathbf{x}}}\frac{3a}{2} - \mathbf{b}_{1,\hat{\mathbf{y}}}\frac{\sqrt{3}a}{2} = 0 \Rightarrow \mathbf{b}_{1,\hat{\mathbf{y}}} = \sqrt{3}\mathbf{b}_{1,\hat{\mathbf{x}}}$$

$$\mathbf{b}_1 \cdot \mathbf{a}_1 = 2\pi \;\Rightarrow\; \mathbf{b}_{1,\hat{\mathbf{x}}}\frac{3a}{2} + \mathbf{b}_{1,\hat{\mathbf{y}}}\frac{\sqrt{3}a}{2} = 2\pi \Rightarrow \mathbf{b}_{1,\hat{\mathbf{x}}} = \frac{2\pi}{3a}, \mathbf{b}_{1,\hat{\mathbf{y}}} = \frac{2\pi}{\sqrt{3}a}$$

Similar derivation for $\mathbf{b}_2$, to yield:

$$\begin{aligned}\mathbf{b}_1 &= \frac{2\pi}{3a}\hat{\mathbf{x}} + \frac{2\pi}{\sqrt{3}a}\hat{\mathbf{y}} \\ \mathbf{b}_2 &= \frac{2\pi}{3a}\hat{\mathbf{x}} - \frac{2\pi}{\sqrt{3}a}\hat{\mathbf{y}}.\end{aligned}$$

To determine the scattering amplitude and phase of a reflected plane wave, the structure factor is very helpful; it is given by $S = \sum_j f_j \exp(-i\mathbf{b}_j \cdot \mathbf{r}_j)$, where $\mathbf{r}_j$ represents the relative position vector of the atoms in the unit cell and f_j the atomic form factor of the carbon atoms.

b) (2 pts) Determine the structure factor of the unit cell of graphene.

(2 pts) The structure factor is given by $S = \sum_j f_j \exp(-i\mathbf{b}_j \cdot \mathbf{r}_j)$, with the form factor f_j for both carbon atoms the same; assume $f_1 = f_2 = f = 1$ in the following.

As indicated in the figure, we now choose the center of the unit cell in the middle of the hexagon. This leads to relative position $\mathbf{r}_j$ for the two atoms at $\left(1/3, 1/3\right)$ and $\left(2/3, 2/3\right)$.

Inserting this into the structure factor yields:

$$S = \exp\left[-\frac{2i\pi}{3}(h+k)\right] + \exp\left[-\frac{4i\pi}{3}(h+k)\right]$$

with h and k the Miller indices. We immediately see that

$$\begin{aligned}S &= 2 \text{ for } h+k = 3n \text{ with } (n = 0, 1, 2, 3...) \\ &= -1 \text{ in all other cases.}\end{aligned}$$

Consider two graphene layers stacked in the so-called Bernal configuration; half of the carbon atoms in the top layer are located exactly above a carbon atom of the bottom layer, whereas the other half are located above the center of the honeycomb cells of the bottom layer). When the top layer is rotated by a small twist angle θ with respect to the bottom layer, twisted bilayer graphene is formed. This structure exhibits a so-called Moiré pattern with periodicity L, which is much larger than the periodicity of a single layer of graphene, as shown in Figure 3.

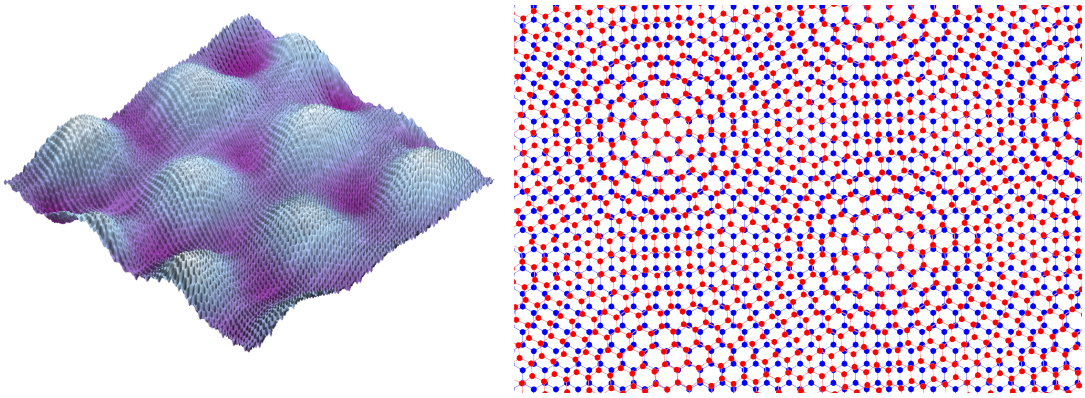
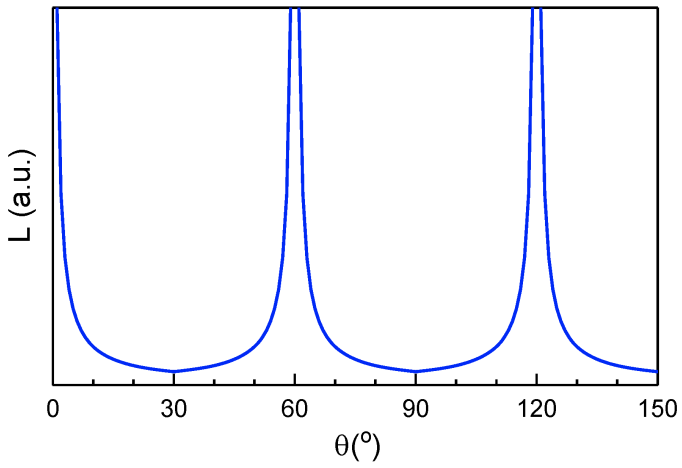


Figure 3: (left) Scanning tunneling microscopy (STM) image of twisted graphene. (right) Schematic Moiré pattern created by rotating two graphene layers (red and blue) relative to each other.

c) (2 pts) Schematically plot the variation of the periodicity L as a function of the twist angle θ.

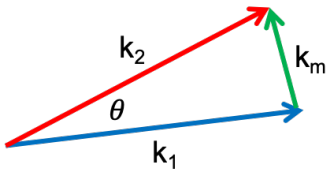
(2 pts) The graph should look something like the figure below. Periodicity should be 60°; L has minima at 30°.



d) (2 pts) Derive an analytical expression for the periodicity L as a function of the twist angle θ.

(2 pts) We consider the periodicity in a single graphene layer to amount to $a\sqrt{3}$, i.e. the lattice parameter; the periodicity of the Moiré pattern is given by L. See figure below; the reciprocal vectors $\mathbf{k_1}$ and $\mathbf{k_2}$ represent the two layers with a relative twist angle θ.

We use that the lengths $|\vec{k_1}| = |\vec{k_2}| = \frac{4\pi}{a\sqrt{3}}$, corresponding to the length of the reciprocal lattice vectors. The Moiré pattern can also be represented by a reciprocal vector $\vec{k_m}$, with $|\vec{k_m}| = \frac{2\pi}{L}$.



Using the cosine rule, we obtain the length of the reciprocal vector $\vec{k_m}$

$$|\vec{k_m}|^2 = |\vec{k_1}|^2 + |\vec{k_2}|^2 - 2|\vec{k_1}||\vec{k_2}|\cos\theta \quad (1)$$

$$\Rightarrow \left(\frac{2\pi}{L}\right)^2 = \left(\frac{4\pi}{a\sqrt{3}}\right)^2 + \left(\frac{4\pi}{a\sqrt{3}}\right)^2 - 2\left(\frac{4\pi}{a\sqrt{3}}\right)^2 \cos\theta \quad (2)$$

$$\Rightarrow L = \frac{a\sqrt{3}}{2\sqrt{2-2\cos\theta}} = \frac{a\sqrt{3}}{4\sin\left(\theta/2\right)} \quad (3)$$

The dispersion relation for electrons in graphene near the Fermi level is given by $E = \hbar v_{\mathrm{F}}|\mathbf{k}|$, where $\hbar$ is the reduced Planck constant, v_{F} the Fermi velocity and $\mathbf{k}$ the momentum of the electrons. The Fermi level can shift due to doping, as schematically shown in Figure 4.

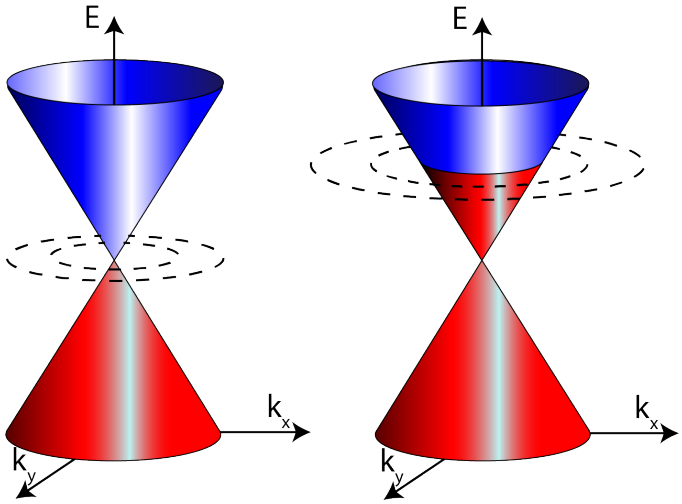


Figure 4: Electron dispersion relation near the Fermi level for undoped (left) and doped (right) graphene.

e) (2 pts) Derive an analytical expression for the charge density in graphene in the case that the Fermi level lies at an energy E_{F} above the Dirac point, also referred to as the charge neutrality point.

(2 pts) From the disperion relation $E = \hbar v_{\mathrm{F}}|\mathbf{k}|$ we obtain $k = |\mathbf{k}| = \frac{E}{\hbar v_{\mathrm{F}}}$ and $\frac{dE}{dk} = \hbar v_{\mathrm{F}}$.

The density of states $g(E)$ relates to that in k-space through, with $g(k) = \left(\frac{1}{2\pi}\right)^2$

$$\begin{aligned} g(E)dE &= g(k)dk && (4)\\ &= \left(\frac{1}{2\pi}\right)^2 \cdot 2\pi k dk \cdot 2 && (5)\\ \Rightarrow g(E) &= \frac{\left(\frac{1}{2\pi}\right)^2 \cdot 2 \cdot 2\pi k}{\frac{dE}{dk}} && (6)\\ &= \frac{1}{\pi}\frac{k}{\hbar v_{\mathrm{F}}} = \frac{E}{\pi\left(\hbar v_{\mathrm{F}}\right)^2} && (7) \end{aligned}$$

where the factor of 2 in eq. 5 is inserted to take into account spin degeneracy. The charge carrier (electron) density is given by

$$n = \int_0^{E_{\mathrm{F}}} 2\frac{E}{\pi\left(\hbar v_{\mathrm{F}}\right)^2} dE = \frac{E_{\mathrm{F}}^2}{\pi\left(\hbar v_{\mathrm{F}}\right)^2} \quad (8)$$

where the factor of 2 reflects the fact that there are two Dirac cones in the Brillouin zone of graphene.

2 Three-level quantum system

Dr. J.W.J. Verschuur
Staff Applied Physics - University of Twente
10 points

Consider a three-level quantum system described by a Hermitian Hamiltonian, that can be written as the sum of two parts:

$$H = H_0 + \lambda H_1$$

where λ is a real number. The eigenstates of H_0 are $|1\rangle$, $|2\rangle$ and $|3\rangle$, with corresponding eigen values, given by:

$$\begin{aligned} H_0 |1\rangle &= 0 \\ H_0 |2\rangle &= \alpha |2\rangle \\ H_0 |3\rangle &= \alpha |3\rangle \end{aligned}$$

This three-level system can be described in matrix notation. The idea is that you obtain a full description of the system from the characteristics given below.

a) (1 pt) Write down H_0 in matrix notation in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis.

(1 pt) This is straight forward, since the three equations in the numeric basis $|i\rangle$ are the eigenvalue equations with the three eigen energies: 0, α and α (degenerate).

b) (2 pts) Write down H_1 in matrix notation, using the lowest amount of unknown different matrix elements (in the same basis). Use symmetry considerations to limit the number of elements in the matrix without losing generality.

(2 pts) A general 3×3 matrix has nine independent coefficients, however the Hamiltonian needs to be Hermitian, *i.e.*:

$$\hat{H}^\dagger = \hat{H}$$

and since $\hat{H}_0$ is diagonal, $\hat{H}_1$ needs to be Hermitian, the off-diagonal coeffcients need to be each others complex conjugates, leading to six real independent coefficients:

$$\hat{H}_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

.

When the (eigenvalue) spectrum of H is computed using perturbation theory, it is found that the eigenstates of H to lowest order in λ are:

$$\begin{aligned} |a\rangle &= |1\rangle \\ |b\rangle &= \frac{1}{\sqrt{2}} \left(|2\rangle + |3\rangle \right) \\ |c\rangle &= \frac{1}{\sqrt{2}} \left(|2\rangle - |3\rangle \right) \end{aligned}$$

and that the corresponding eigenvalues are:

$$\begin{aligned} E_a &= -\frac{\lambda^2}{\alpha} + \mathcal{O}(\lambda^3), \\ E_b &= \alpha + \lambda + \frac{\lambda^2}{\alpha} + \mathcal{O}(\lambda^3), \\ E_c &= \alpha - \lambda + \mathcal{O}(\lambda^3). \end{aligned}$$

c) (4 pts) Determine as many of the matrix elements of H_1 as you can.

(4 pts) Since the energies are given in the alpha-numeric basis, it is most convenient to determine $\hat{H}_1$ first in the alpha-numeric basis. From the given energies – sorting by order of λ – we have the following three sets of equations:
For $\hat{H}_0$:

$$\begin{aligned} E_a^{(0)} &= \langle a|\hat{H}_0|a\rangle = 0 \\ E_b^{(0)} &= \langle b|\hat{H}_0|b\rangle = \alpha \\ E_c^{(0)} &= \langle c|\hat{H}_0|c\rangle = \alpha \end{aligned}$$

For $\hat{H}_1$, first order in λ:

$$\begin{aligned} E_a^{(1)} &= \langle a|\hat{H}_1|a\rangle = 0 \\ E_b^{(1)} &= \langle b|\hat{H}_1|b\rangle = \lambda \\ E_c^{(1)} &= \langle c|\hat{H}_1|c\rangle = -\lambda \end{aligned}$$

this results in:

$$\begin{aligned} a_{11} &= 0 \\ a_{22} &= \lambda \\ a_{33} &= -\lambda \end{aligned}$$

For $\hat{H}_1$, second order in λ:

$$\begin{aligned} E_a^{(2)} &= \frac{\left|\langle b|\hat{H}_1|a\rangle\right|^2}{E_a^{(0)} - E_b^{(0)}} + \frac{\left|\langle c|\hat{H}_1|a\rangle\right|^2}{E_a^{(0)} - E_c^{(0)}} = -\frac{\lambda^2}{\alpha} \\ E_b^{(2)} &= \frac{\left|\langle a|\hat{H}_1|b\rangle\right|^2}{E_b^{(0)} - E_a^{(0)}} + \frac{\left|\langle c|\hat{H}_1|b\rangle\right|^2}{E_b^{(0)} - E_c^{(0)}} = \frac{\lambda^2}{\alpha} \\ E_c^{(2)} &= \frac{\left|\langle a|\hat{H}_1|c\rangle\right|^2}{E_c^{(0)} - E_a^{(0)}} + \frac{\left|\langle b|\hat{H}_1|c\rangle\right|^2}{E_c^{(0)} - E_b^{(0)}} = 0 \end{aligned}$$

which is only valid in case the unperturbed energies are different. Since $E_c^{(0)} = E_b^{(0)}$, we must have:

$$\langle b|\hat{H}_1|c\rangle = \langle c|\hat{H}_1|b\rangle = a_{23} = 0$$

From this we obtain:

$$\langle a|\hat{H}_1|c\rangle = \langle c|\hat{H}_1|a\rangle = a_{13} = 0$$

and since:

$$E_a^{(0)} - E_b^{(0)} = \alpha$$

we obtain:

$$\langle a|\hat{H}_1|b\rangle = \langle b|\hat{H}_1|a\rangle = a_{12} = \pm\lambda$$

Which results in a perturbation Hamiltonian $\hat{H}_1$ in the alpha-numeric basis:

$$\hat{H}_1^{(\alpha)} = \begin{pmatrix} 0 & \pm\lambda & 0 \\ \pm\lambda & \lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}$$

.

d) (2 pts) Two different bases are used in this problem, the numeric basis and the alpha-numeric basis. Give the transformation, that relates the two bases.

(2 pts) The transformation matrix for the transformation between the numeric and alpha-numeric basis can be derived directly from the given relations:

$$|\alpha n\rangle = \mathbf{T}\,|n\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} |n\rangle$$

or the back transformation:

$$|n\rangle = \mathbf{T}^{-1}\,|\alpha n\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} |\alpha n\rangle$$

.

e) (2 pts) Use the transformation to represent the Hamiltonian H in both representations.

(2 pts) In order to transform the perturbation Hamiltonian $\hat{H}_1$ from the alpha-numeric basis to the numeric basis we use:

$$|n'\rangle = \mathbf{T}^{-1}\hat{H}_1^{(\alpha)}\,|\alpha n\rangle = \mathbf{T}^{-1}\hat{H}_1^{(\alpha)}\mathbf{T}\,|n\rangle = \hat{H}^{(n)}\,|n\rangle$$

Using the matrices to obtain the perturbation Hamiltonian in the numeric basis we obtain:

$$\begin{aligned} \hat{H}_1^{(n)} &= \mathbf{T}^{-1}\hat{H}_1^{(\alpha)}\mathbf{T}\,|n\rangle \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & \pm\lambda & 0 \\ \pm\lambda & \lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \pm\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} & 0 & \lambda \\ \pm\frac{1}{\sqrt{2}} & \lambda & 0 \end{pmatrix} \end{aligned}$$

So the perturbation Hamiltonian in both bases:

$$\hat{H}_1^{(\alpha)} = \begin{pmatrix} 0 & \pm\lambda & 0 \\ \pm\lambda & \lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}$$

$$\hat{H}_1^{(n)} = \begin{pmatrix} 0 & \pm\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} & 0 & \lambda \\ \pm\frac{1}{\sqrt{2}} & \lambda & 0 \end{pmatrix}$$

.

f) (1 pt) Is either one of the two bases an eigen-system for the Hamiltonian H?

(1 pt) No. The total Hamiltonian in the alpha-numeric basis is:

$$\hat{H}_0^{(\alpha)} + \hat{H}_1^{(\alpha)} = \begin{pmatrix} 0 & \pm\lambda & 0 \\ \pm\lambda & \alpha+\lambda & 0 \\ 0 & 0 & \alpha-\lambda \end{pmatrix}$$

and the Hamiltonian in the numeric basis is:

$$\hat{H}_0^{(n)} + \hat{H}_1^{(n)} = \begin{pmatrix} 0 & \pm\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} & \alpha & \lambda \\ \pm\frac{1}{\sqrt{2}} & \lambda & \alpha \end{pmatrix}$$

Both representation have a total Hamiltonian that is not on diagonal form in either one of the two bases, so the corresponding eigen-vectors deviate from the numeric and the alpha-numeric basis sets. Calculating the eigen-values in both representations give the same values. The corresponding eigen-vectors, are different, since they are defined with respect to different bases.

Perturbation theory

For your convenience, the expressions to calculate the corrections in first and second order respectively, are given by:

$$\begin{aligned} E_i^{(1)} &= \langle i|H_1|i\rangle \\ E_i^{(2)} &= \sum_{j\neq i}^{N} \frac{\left|\langle j|H_1|i\rangle\right|^2}{E_i^0 - E_j^0} \end{aligned}$$

The first-order correction to the wave function is given by:

$$|i\rangle^{(1)} = \sum_{j\neq i}^{N} \langle j|H_1|i\rangle\,|j\rangle$$

Note that these expression only hold for non-degenerate perturbation theory. Degenerate perturbation, if applicable, is part of the problem.

3 Putting equilibrium fluctuations to work

Dr. K.H. Mathwig and Prof. Dr. S.J.G. Lemay
Bio Electronics - University of Twente
10 points

In nanofluidics, a fluid is confined to sub-micron length scales in one or more dimension. This can cause effects which are normally too small to be noticed to become dominant. Figure 5 shows a sketch of a lithographically fabricated nanofluidic device. It consists of a nanochannel of height ∼100 nm and width 5 μm. Electrode pairs of length $L = 50$ μm are embedded in its floor and ceiling, which allows exchanging electrons with molecules in the fluid via electrochemical reactions. This can be employed to create extremely sensitive detectors via so-called redox cycling. In this method, potentials are applied to the electrodes such that the floor of the channel can donate electrons to electrochemically active molecules that come within the electron tunneling distance (∼1 nm) while the ceiling can similarly accept electrons. Molecules undergoing Brownian motion in the nanochannel repeatedly collide with the electrodes, ferrying electrons from floor to ceiling and thus generating a measurable electrical current. Because of the small distance between the electrodes, this process is very efficient and a single molecule ferries about 10^5 electrons/second.

Here we consider a simple model of mass transport in this system. Because of confinement in the width and height directions, we only consider a single dimension, x, the position along the length of the channel. The measured current is proportional to the number of molecules present in the detection volume between the electrodes ($0 < x < L$); for simplicity we take this number $N(t)$ to represent the measured signal. Molecules can either diffuse or undergo convection along the channel such that the local concentration $c(x,t)$ is given by the drift-diffusion equation,

$$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2} - v\frac{\partial c(x,t)}{\partial x}. \tag{9}$$

Here D is the diffusion coefficient for the molecules (a constant) and v is the advection velocity of the fluid in the x direction. We do not worry about the inlet or outlet regions, considering the channel to be infinite in the $-x$ and $+x$ directions.

The average number of molecules, $\bar{N}$, is simply given by the concentration of molecules in the fluid times the volume of the detection region between the electrodes. However, molecules can enter and leave the detection volume by Brownian motion. Since molecules diffuse independently of each other, this causes $N(t)$ to fluctuate randomly in time around $\bar{N}$, conveniently written as $N(t) = \bar{N} + \Delta N(t)$. Such fluctuations are universal for systems in diffusive equilibrium, but usually they are too small to be detected (more precisely, $\Delta N/\bar{N}$ vanishes in the thermodynamic limit). Because the present system is very small, however (volume 25 femtoliters), $\bar{N}$ is small and the fluctuations $\Delta N(t)$ become observable. In fact, these fluctuations are the dominant source of noise in sensors based on this device, as illustrated in Figure 5(b). Here we look further into these fluctuations based on the 1D model introduced above.

a) (2 pts) First consider a single pair of facing electrodes. We first focus on the purely diffusive case ($v = 0$). Suppose that a single molecule starts at $t = 0$ somewhere in the detection region, $0 < x < L$. As time passes, the chance that the molecule is still in the detection region diminishes. Give an expression for the characteristic time scale over which this process occurs. Justify your answer in 1-3 sentences and equations/diagrams as needed.

(2 pts) The characteristic time for diffusing a distance L is $L^2/2D$, as can be deduced (within the factor of 2) from dimensional analysis.

b) (4 pts) More quantitatively, we define the autocorrelation function,

$$C_{11}(\tau) = \left\langle \Delta N(t) \Delta N(t+\tau) \right\rangle, \tag{10}$$

where the brackets represent an average over time. Evaluate $C_{11}(\tau)$ for this system. We are only interested in the τ dependence; you need not normalize your result.

(4 pts) If we are not interested in the absolute normalization, we can perform the calculation for a single particle starting in the detection region. The probability of finding the molecule inside the detection volume after time $\tau$is given by

$$\int_0^L n(x_2, \tau) dx_2,$$

where x_2 represents the position at time τ. Summing over all possible initial positions x_1 at $t = 0$ gives the autocorrelation function,

$$\begin{aligned}
\int_0^L dx_1 \int_0^L dx_2 n(x_2, \tau) &= \int_0^L dx_1 \int_0^L dx_2 \frac{1}{\sqrt{4\pi D\tau}} e^{-(x_2-x_1)^2/4D\tau} \\
&= \int_0^L dx_1 \int_{-x_1/\sqrt{4D\tau}}^{(L-x_1)/\sqrt{4D\tau}} du \frac{e^{-u^2}}{\sqrt{\pi}} \\
&= \frac{1}{2} \int_0^L dx_1 \left[\operatorname{erf}\left(\frac{L-x_1}{\sqrt{4D\tau}} \right) - \operatorname{erf}\left(\frac{-x_1}{\sqrt{4D\tau}} \right) \right] \\
&= L \operatorname{erf}\left(\frac{L}{\sqrt{4D\tau}} \right) - \sqrt{\frac{4D\tau}{\pi}} \left[1 - e^{-L^2/4D\tau} \right]
\end{aligned}$$

c) (1 pt) Suppose that we now have two detectors, each of length L, localized along the length of the nanochannel and separated by a gap of lenth g. Gap 1 is upstream and gap 2 is downstream. In the presence of a finite advection velocity $v > 0$, number fluctuations at the upstream detector can propagate to the downstream detector. If this takes too long, however, Brownian motion will wash out the fluctuations before they can reach the second detector. Assuming that $g \gg L$, give an estimate of the minimum characteristic velocity v required for fluctuations to be preserved between detectors. Justify your answer in 1-3 sentences and equations/diagrams as needed.

(1 pt) Concentration fluctuations over a length scale L reorganize in time $\tau \simeq L^2/D$. To preserve fluctuations the transit time between the electrodes must be shorter than this time. That is, v must satisfy $g/v \ll \tau$ or $v \gg gD/L^2$.

d) (3 pts) We now define the crosscorrelation function,

$$C_{12}(\tau) = \left\langle \Delta N_1(t) \Delta N_2(t+\tau) \right\rangle, \tag{11}$$

where $\Delta N_1(t)$ and $\Delta N_2(t)$ are the simultaneously measured fluctuations at the upstream and downstream detectors, respectively. Derive an expression for $C_{12}(\tau)$, again disregarding normalization.

(3 pts) The calculation is analogous to that in (b) except that now each particle drifts with velocity v such that

$$n(x_2, t) = \frac{1}{\sqrt{4\pi D t}} e^{-(x_2 - x_1 - v\tau)^2/4Dt}.$$

It follows that the cross-correlation function is

$$\begin{aligned}
C_{12}(\tau) &= \int_0^L dx_1 \int_{L+g}^{2L+g} dx_2 n(x_2, \tau) \\
&= \int_0^L dx_1 \int_{L+g}^{2L+g} dx_2 \frac{1}{\sqrt{4\pi D\tau}} e^{-(x_2 - x_1 - v\tau)^2/4D\tau} \\
&= \int_0^L dx_1 \int_{(L+g-x_1-v\tau)/\sqrt{4D\tau}}^{(2L+g-x_1-v\tau)/\sqrt{4D\tau}} du \frac{e^{-u^2}}{\sqrt{\pi}} \\
&= \frac{1}{2}\left[(2L+g-v\tau)\,\mathrm{erf}\left(\frac{2L+g-v\tau}{\sqrt{4D\tau}}\right) + (g-v\tau)\,\mathrm{erf}\left(\frac{g-v\tau}{\sqrt{4D\tau}}\right) \right. \\
&\quad \left. -2(L+g-v\tau)\,\mathrm{erf}\left(\frac{L+g-v\tau}{\sqrt{4D\tau}}\right) \right] \\
&\quad -\sqrt{\frac{D\tau}{\pi}}\left[2e^{-(L+g-v\tau)^2/4D\tau} - e^{-(2L+g-v\tau)^2/4D\tau} - e^{-(g-v\tau)^2/4D\tau} \right].
\end{aligned}$$

Note that the answer to (b) can be recovered by setting $g = -L$ and $v = 0$ in this expression.

The cross-correlation function can be used to infer the advection velocity of the fluid in the nanochannel, as illustrated in Figure 6 for the case $g \ll L$. This is not a trivial achievement, as flow rates in these nanochannels are only a few drops per year! Figure 6(c) is the result that you are asked to derive in part (d).

This question is based on K. Mathwig, D. Mampallil, S. Kang and S. G. Lemay, Phys. Rev. Lett. **109**, 118302 (2012).

Some useful mathematical details:

1. The solution to Equation 9 for a single particle starting at the origin ($c(x,0) = \delta(x)$) is

$$c(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-x^2/4Dt}, \tag{12}$$

where $c(x,t)$ is the probability of the particle to be present at x at time t.

2. You may find it convenient to use the shorthand notation for the error function,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du, \tag{13}$$

where $\mathrm{erf}(0) = 0$, $\mathrm{erf}(\infty) = 1$, $\mathrm{erf}(-x) = -\mathrm{erf}(x)$ and

$$\int \mathrm{erf}(x) dx = x\,\mathrm{erf}\,x + \frac{e^{-x^2}}{\sqrt{\pi}} \tag{14}$$

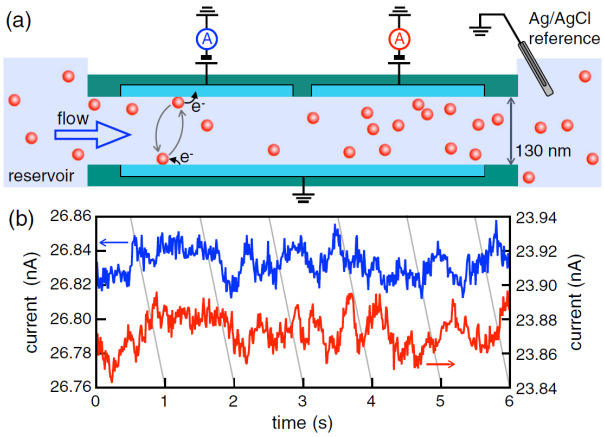


Figure 5: (a) Schematic of the concept for measuring fluctuations in the number of electrochemically active molecules ($N(t)$, red balls) between two electrodes in a nanochannel. (b) Raw current-time traces recorded at two electrode pairs located downstream from each other. The noise is due to statistical fluctuations in $N(t)$. These data were obtained at a flow rate corresponding to a transit time between the electrode pairs of approximately 0.2 s. Gray lines are guides to the eye to highlight correlations between the two traces.

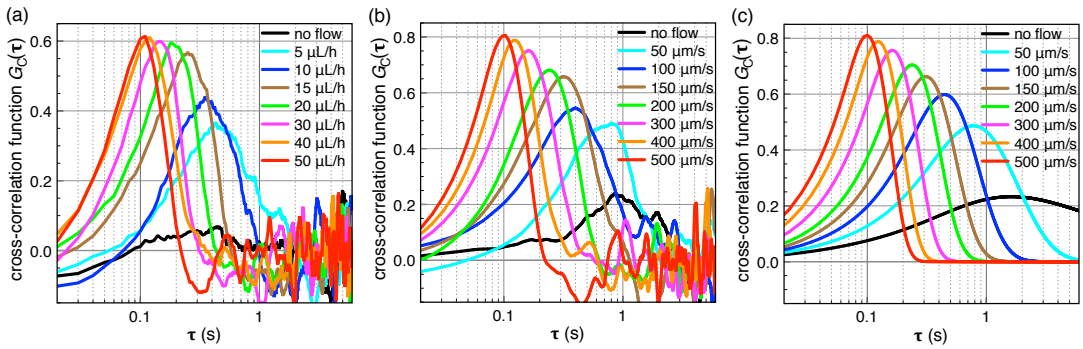


Figure 6: (a) Cross-correlation functions of current-time traces recorded experimentally at the two electrode pairs ($L = 50$ μm, $g = 2$ μm) for different pump flow rates. (b) Cross-correlation functions obtained from a one-dimensional random walk simulation of Equation (1). The noise in the traces at large τ is comparable to that in panel (a) because the simulation was performed over the same time interval. (c) Analytically derived cross-correlation function. This is a plot of the answer to part (d) of this question.

4 Infinite Potential

Dr. Ir. H.L. Offerhaus
Optical Sciences - University of Twente
10 points

Estimate the magnetic vector potential $\mathbf{A}$ inside a cylinder of radius $R = 5mm$ and length $L = 50mm$ that has a current I flowing longitudinally over the surface which is uniformly distributed over the circumference. As $R << L$, the cylinder can be considered to be thin and long.

Hint: Recall that $\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}}{r}\,dV$.

The magnetic vector potential $\mathbf{A}$ is given by:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}}{r}\,dV. \tag{15}$$

Note that it follows that $\mathbf{A}$ has only a component in z-direction, as $\mathbf{A}$ is parallel to $\mathbf{J}$ which is also parallel to the current $\mathbf{I}$. From Maxwell it follows that

$$\oint \mathbf{H}\cdot d\mathbf{l} = I_{\text{encl}}. \tag{16}$$

Furthermore, the following relationships hold:

$$\mathbf{B} = \mu\mathbf{H}, \tag{17}$$

$$\mathbf{B} = \nabla\times\mathbf{A}. \tag{18}$$

From Eq. ? it follows that within the cylinder $\mathbf{H} = \mathbf{0}$ as $I_{\text{encl}} = 0$, such that from Eq. 17 and 18 it follows that $\mathbf{B} = \mathbf{0}$ and thus $\nabla\times\mathbf{A} = \mathbf{0}$. This means that $\mathbf{A}$ can be written as the divergence of some scalar field, i.e. $\mathbf{A} = \nabla\phi$.

As $\mathbf{A}$ has only a non-zero component in z-direction and the hint was given that the cylinder can be considered to be small, the variation in z-direction must be small such that we can assume that $\mathbf{A}$ = const. within the cylinder.

As $\mathbf{A}$ = const. within the cylinder, we'll place the origin of our cylindrical coordinate system in the middle of the cylinder such that we have axissymetry and the current is located at distance R. Then, we've got according to Eq. 15:

$$\mathbf{A} = \hat{z}\frac{\mu_0}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{I}{\sqrt{z^2+R^2}}\,dz. \tag{19}$$

It follows that $z = R\tan\theta$, $dz = \frac{R}{\cos\theta^2}d\theta$ and $r = \frac{R}{\cos\theta}$.

From Eq. 19 it follows that:

$$\begin{aligned}\mathbf{A} &= \hat{z}\frac{\mu_0}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{I}{\sqrt{z^2+R^2}}\,dz \\ &= \hat{z}\frac{\mu_0 I}{4\pi}\int_{-\theta_{max}}^{\theta_{max}}\frac{1}{\cos\theta}\,d\theta \\ &= \hat{z}\frac{\mu_0 I}{4\pi}[\ln\frac{1}{\cos\theta}+\tan\theta]_{-\theta_{max}}^{\theta_{max}} \\ &= \hat{z}\frac{\mu_0 I}{2\pi}\ln\frac{1}{\cos\theta_{max}}+\tan\theta_{max}\end{aligned} \tag{20}$$

since both are anti-symmetric functions of θ.

Also, $\tan\theta_{max} = \frac{L}{2R}$ and $\cos\theta_{max} = \frac{R}{\sqrt{R^2+\frac{L}{2}^2}} = \frac{2R}{L\sqrt{\frac{2R}{L}^2+1}} \approx \frac{2R}{L(1+\frac{2R^2}{L^2})} \approx \frac{2R}{L}(1-\frac{2R^2}{L^2}) \approx \frac{2R}{L}$, i.e. $\frac{1}{\cos\theta_{max}} \approx \frac{L}{2R}$. This gives according to Eq. 20:

$$\begin{aligned}\mathbf{A} &= \hat{z}\frac{\mu_0 I}{2\pi}\ln\frac{L}{R} \\ &= \hat{z}\frac{\mu_0 I}{2\pi}\ln 10 \\ &= \hat{z}\frac{\mu_0 I}{2\pi} * 2.3 \\ &= \hat{z}0.366\mu_0 I.\end{aligned} \tag{21}$$

(3 pts) Realising $\mathbf{A}$ is homogeneous such that you only have to calculate the value of $\mathbf{A}$ in the centre of the cylinder.
(3 pts) Finding the integral in the centre (doesn't matter if integral is given in angles or in terms of x, z).
(2 pts) Finding the integrand.
(2 pts) Completing the answer correctly.

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APPLIED PHYSICS

5 Touching Pulses

Dr. Ir. H.L. Offerhaus
Optical Sciences - University of Twente
10 points

Given is the following expression for intramodal dispersion of refractive index n_1 for a single mode through a fiber as:

$$\lambda^2 \frac{d^2 n_1}{d\lambda^2} = 0.025$$

and a laser that produces pulses of 20 fs duration ($1/e$ width) centered around a wavelength of 850 nm with a repetition frequency of 100 MHz.
How many km's of fiber do you need to let the tail ($1/e$ point) of one pulse be touched by the front ($1/e$) of the next pulse?

Hint: Recall that the group velocity v_g is given by $v_g = \frac{c}{n_1 - \lambda \frac{dn_1}{d\lambda}}$.

The pulse spread is due to the second order dispersion of a material which is the derivative of the group velocity. The 1/group velocity is also known as τ_g, which is given by:

$$\tau_g = \frac{1}{c}(n_1 - \lambda \frac{dn_1}{d\lambda}). \tag{22}$$

Thus, for a fiber of length L the total pulse delay τ_p is given by $\tau_p = L\tau_g$.

To find the pulse spreading σ_p, we make a Taylor expansion to find:

$$\begin{aligned} \sigma_p &= L\sigma_g \\ &= L\frac{d\tau_g}{d\lambda}\sigma_\lambda. \end{aligned} \tag{23}$$

We note that

$$\begin{aligned} \frac{d\tau_g}{d\lambda} &= \frac{\lambda}{c}(\frac{dn_1}{d\lambda} - \frac{d^2 n_1}{d\lambda^2} - \frac{dn_1}{d\lambda}) \\ &= -\frac{\lambda}{c}\frac{d^2 n_1}{d\lambda^2}, \end{aligned} \tag{24}$$

such that

$$\begin{aligned} \sigma_p &= \frac{\sigma_\lambda L}{c}|\lambda \frac{d^2 n_1}{d\lambda^2}| \\ &= \frac{\sigma_\lambda L}{c\lambda}|\lambda^2 \frac{d^2 n_1}{d\lambda^2}|. \end{aligned} \tag{25}$$

Furthermore,

$$\begin{aligned}\sigma_\lambda &= \frac{\Delta f}{f}\lambda \\ &= \frac{\lambda^2}{c\tau} \\ &= \frac{(850 * 10^{-9})^2}{3 * 10^8 * 20 * 10^{-15}} \\ &= 120 * 10^{-9} m.\end{aligned} \tag{26}$$

This gives

$$\begin{aligned}\frac{\sigma_p}{L} &= \frac{\sigma_\lambda}{c\lambda}|\lambda^2 \frac{d^2 n_1}{d\lambda^2}| \\ &= \frac{120 * 10^{-9}}{3 * 10^8 * 850 * 10^{-9}} * 0.025 \\ &= 1.176 * 10^{-11} s/m \\ &= 1.176 * 10^{-8} s/km.\end{aligned} \tag{27}$$

As the repetition frequency is 100 MHz, this means that there is $1 * 10^{-8}$ s between two successive pulses. This means that the needs to be $1 * 10^{-8}/(1.176 * 10^{-8}) = 0.85$ km fiber length, i.e. slightly less than one km fiber.

(3 pts) Using the concept of group dispersion delay (or a similar quantity).
(3 pts) Using a Taylor expansion.
(2 pts) Calculating the bandwidth of the pulse.
(2 pts) Completing the answer correctly.

6 The Clements Decomposition

Dr. J.J. Renema
Complex Photonic Systems - University of Twente
10 points

In quantum optics, we are quite frequently interested in making a set of optical modes interfere in a controlled fashion. Such control is particularly important when performing quantum computations with optical systems, since the way the optical modes interfere controls the computation which is being performed. One way to do this is in photonic chips, where a series of waveguides is written in some dielectric medium. Light is then typically sent into the side of the chip via optical fibers, and the chip then functions as a kind of shunting yard for light.

In this exercise, we will explore some of the quantum mechanics of the process of programming such a chip. In particular, we are interested in how many resources (optical elements) are required to produce controlled interference of N distinct optical modes.

If an optical system is lossless, the transformation on the optical modes is given by a unitary matrix U, in the form of $E_{out} = UE_{in}$, where E is the vector of electric field amplitudes.

a) (2 pts) Explain why the fact that we require the system to be lossless means that the number of input modes has to be equal to the number of output modes, i.e. why U has to be a square matrix. *Hint: keep in mind that linear optical systems must obey reciprocity.*

(2 pts) If the number of optical modes at the input and output were unequal, one can ask what happens if you shine light at each of the larger set of modes consecutively. Either light from two modes must be routed to the same mode, which contradicts reciprocity, or some light must disappear, which contradicts losslessness. We therefore conclude that a lossless N to M transformation is impossible for $N \neq M$.

Recall that a unitary matrix is defined by the relation $U^\dagger U = I$, i.e. $(U_i, U_j) = \delta_{ij}$, where $(,)$ is the inner product and U_i is the i-th column of U.

b) (2 pts) Explain why the lossless character of the transformation implies that U must not just be square, but also unitary.

(2 pts) Unitary matrices preserve the vector norm, and since $|E|^2$ is the energy density, this is equivalent to the requirement that there is no optical loss.

We would like to have the power to program out arbitrary unitary optical transformations. Surprisingly, this can be done with a relatively modest number of optical components. To guess how many components we need, we will count the number of free parameters in a unitary transformation.

c) (2 pts) Compute the number of real free parameters in a unitary matrix. Hint: begin with an arbitrary complex matrix, and consider the number of restrictions which unitarity imposes. Furthermore, consider that since we will not re-interfere the light after the unitary transformation, the phase of the light at the output of the system doesn't matter, which further reduces the number of free parameters by N.

(2 pts) An arbitrary complex matrix has N^2 independent entries, which each have an amplitude and phase, for $2N^2$ real degrees of freedom. The unit norm of the columns removes N of these (note that (x, x) is always real for any x), and orthogonality removes $N(N-1) = N^2 - N$ (since in the inner product we have to make both real and compex part zero). Finally, the phase insensitivity requirement removes a further N. This leaves us with $2N^2 - (N^2 - N) - 2N = N^2 - N$ degrees of freedom.

The elementary component which we have at our disposal is an arbitrary 2-by-2 optical transformation, composed of a single beam splitter and a phase shifter.

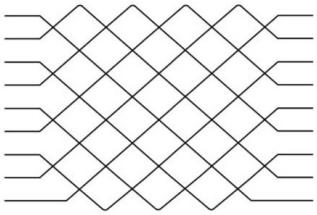


Figure 7: The Clements decomposition. Each line represents an optical mode, and each crossing point between two modes represents an arbitrary 2-mode transformation.

d) (2 pts) Show that the number of such components required is $N(N-1)/2$

(2 pts) Using the formula for $N = 2$ gives two degrees of freedom per 2-by-2 transformation, hence you need $N(N-1)/2$ beam splitters.

e) (1 pt) What physical interpretation can you give to this number?

(1 pt) This is the total number of pairs which you can make with N modes, which means that each mode needs to interfere with every other mode exactly once in order to do an arbitrary transformation.

That this number of beam splitters and phase shifters is indeed sufficient to implement an arbitrary optical transformation was rigorously proven by (then PhD student) William Clements in 2016. His scheme has the particularly attractive feature that it only requires transformations between adjacent modes, which means that it can be implemented in a planar chip, i.e. without optical modes going out of plane, which significantly simplifies fabrication. A picture of the scheme is given in Figure 7.

f) (1 pt) Give a simple physical argument why deleting a single row of beam splitters from this scheme would make it no longer universal.

(1 pt) If you would delete one row, light from the top wouldn't be able to reach the bottom, and vice versa. Therefore, any transformation requiring this cannot be implemented and the scheme would not be universal.

7 Fast spinning with electrons

Dr. Ir. M. Veldhorst and Ir. N. Hendrickx
QuTech - Delft University of Technology
15 points

In quantum information it is crucial that information can be stored on quantum bits (qubits) for a long time. In quantum computation with silicon quantum dots, single electrons are confined with electric gates and the spin states are used to define qubits. An important time scale is the decay time over which an electron in the spin up state relaxes to the spin down state. This time can be measured by preparing a spin down electron, applying a π-pulse using electron-spin-resonance (ESR) such that the electron spin evolves to the spin up state, and by measuring the probability of the electron spin being in the spin up state as a function of waiting time. The measurement can be done using a step called spin-to-charge conversion, see the drawing below, where nearby charge sensors are used to detect tiny changes in the capacitance.

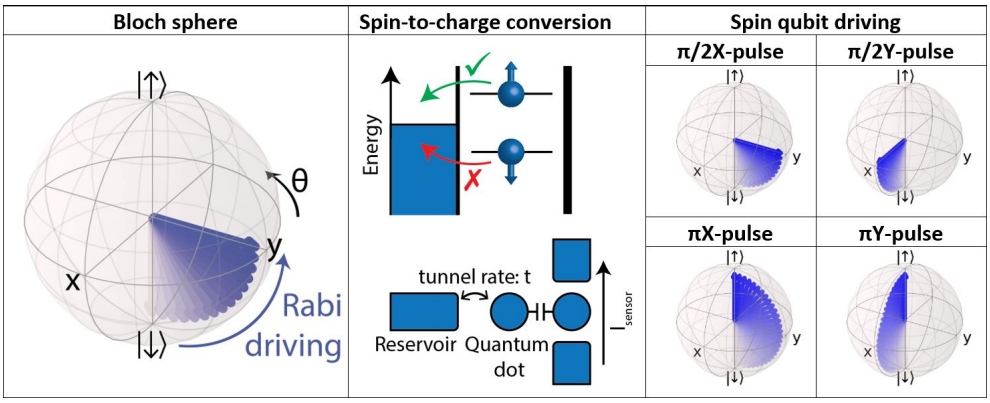


Figure 8: The spin qubit quantum toolbox.

Readout

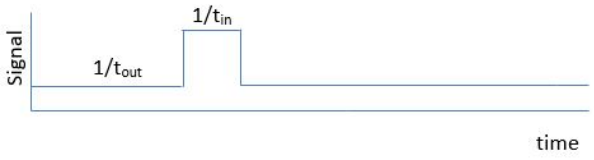
a) (1 pt) Draw the signal of the charge sensor when a quantum dot is filled with an electron in the spin down state and tuned to the readout position. Include labels associated with relevant time scales.

A straight line – no variation in signal is detected (up to noise). A spin down electron cannot tunnel out and remains in the dot.

(1 pt) for correct drawing.

b) (2 pts) Draw the signal of the charge sensor when a quantum dot is filled with an electron in the spin up state and tuned to the readout position. Include labels associated with relevant time scales.

A line with a 'bump' or 'dip', corresponding to an electron in the spin up state going out and an electron in the spin down state going in.



(1 pt) for correct drawing.
(1 pt) for recognizing that the time is dependent on 1/t (t or t_{out} and t_{in} are both correct).

c) (1 pt) Explain how readout leads to spin initialization and give the initialized spin.

The end state after readout is always spin down. From answer a and b one can see that the final signal always corresponds to a dot filled with an electron in the spin down state.

(1 pt) for correct answer.

Qubit operation

The qubit state can be changed by applying well calibrated ESR pulses, such that the spin rotates over a certain angle around a certain axis. Four different pulses that can be applied are given in Figure 8.

d) (1 pt) How can we turn off spin-to-charge conversion during these operations?

There are two possibilities: the energy can be changed such that no electron can tunnel out. Alternatively, the tunnel coupling t can be reduced such that no electron can tunnel out.

(1 pt) for a correct answer.

e) (2 pts) Describe how the experiment to measure the spin life time needs to be modified in order to measure the dephasing time of a single spin.

Instead of a π-pulse, one must apply a $\pi/2$ pulse before the waiting time and a $\pi/2$ after the waiting time. The first pulse is there to prepare the electron spin in a superposition state. The second pulse is there to return to an eigenstate that can be measured.

(1 pt) if the answer only mentions the first $\pi/2$ pulse.
(2 pts) if both $\pi/2$ pulses are mentioned.

A silicon spin qubit is prepared pointing along the positive x-axis direction. In practical experiments, qubits are subject to several types of noise. Consider constant magnetic noise, such that the actual field is 1 nT lower than targeted. Due to the slower Larmor precession, the qubit acquires a phase shift over time as compared to the targeted precession.

f) (1 pt) Show how one can calculate the acquired phase shift over time.

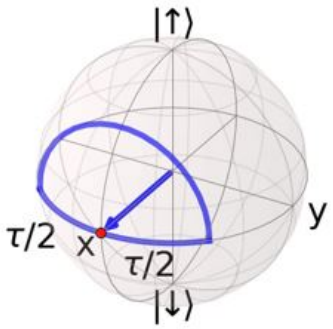
The Larmor frequency is calculated by $\nu = \gamma B/2$, with γ the gyromagnetic ratio. The reduction in precession frequency is thus $\delta\nu = \gamma\delta B$, with $\delta B = 1\text{nT}$. Alternatively, one can also express in terms of the bohr magneton and g-factor: $\delta\nu = g u_B \delta B/\hbar$. The time dependence of the phase shift is determined by the frequency, $\delta\theta = \sin\delta\nu t$).

(1 pt) for correct answer.

Assume for the following two questions that driving takes zero time and thus time needs to be spend precessing.

g) (3 pts) Design a sequence consisting of a single pulse for a given sequence length τ such that the qubit (prepared pointing in the positive x-axis direction) has acquired zero phase shift compared to the situation if there was no constant noise. Write down the sequence and include the length of each segment. Also draw on the Bloch sphere the spin evolution due to the pulses and the noise.

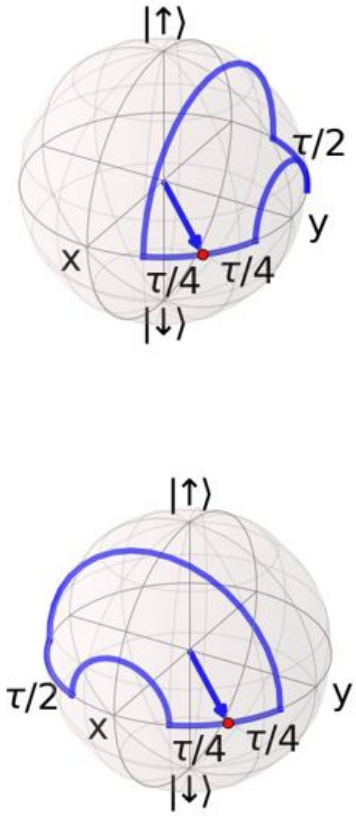
The sequence consist of first waiting a $\tau/2$, followed by applying a Y pulse, and waiting again a time $\tau/2$. Due to the constant noise, the qubit acquires a phase shift , but this shift is exactly cancelled due to the application of the Y-pulse and waiting the same time again. This can be seen in Bloch sphere drawing below.



(1 pt) for writing sequence $\tau/2$ - π pulse - $\tau/2$.
(2 pts) for writing sequence $\tau/2$ – Y (π pulse along y) - $\tau/2$.
(1 pt) for correct drawing Bloch sphere.

h) (2 pts) Now assume the qubit is initialized at $\theta = \frac{\pi}{4}$ from the x-axis. Give the simplest pulse sequence for a given sequence length τ such that the qubit has acquired zero phase shift compared to the situation if there was no noise. Draw on the Bloch sphere the spin evolution due to the pulses and the noise.

The sequence is now $\tau/4$ – X - $\tau/4$ - $\tau/4$ – X - $\tau/4$. Alternatively: $\tau/4$ – Y - $\tau/4$ - $\tau/4$ – Y - $\tau/4$.



(1 pt) for correct sequence.
(1 pt) for correct Bloch sphere.

Instead of a constant magnetic noise term, the spin qubit is now subject to a completely randomly fluctuating magnetic field.

i) (1 pt) Can you still define a pulse sequence such that for a given time τ the qubit has acquired zero phase shift compared to its initial state? Explain why.

No. There are parts of the noise that are uncorrelated and this cannot be correct (in particular, the high frequency part cannot be perfectly corrected).

(1 pt) for correct answer.

In experiments, one encounters often a noise spectrum with a 1/f spectral density, which is believed to originate from defects in the substrate close to the spin qubits. Quantum operations in experiments take a finite time and we thus strive to maximize the number of operations within the dephasing time to do meaningful quantum calculations.

j) (1 pt) If we can find a method to double the speed at which we can do qubit operations, can we do more, less, or exactly two times as many rotations? Explain why.

We can do more than two times as many rotations. One may think that we can do twice as many rotations due to the two times faster driving. However, since we are continuously doing rotations, components of the noise with a frequency lower than the driving frequency will be cancelled (as observed in question g and h), thereby increasing the coherence time. Thus, by driving faster, we can extend the coherence time more and thus do more than two times the number of rotations.

(1 pt) for correct answer.

8 Pointy Ice Drops

Prof. Dr. J.H. Snoeijer, Dr. A. Marin, Dr. S.G. Huisman
Physics of Fluids - University of Twente
10 points

When a drop of liquid water is placed on a cold surface, it freezes into a singular shape. The initially rounded liquid drop is transformed into a pointy ice drop, which at its tip exhibits a conical shape. This freezing process can be seen in the photographs of Figure 9(ab). The water drops are initially at room temperature, while the surface is at a temperature that can range from $-10°$C to $-40°$C. The key ingredient behind this phenomenon is that the density of solid ice (ρ_s) is lower than the density of liquid water (ρ_ℓ). We therefore introduce the density ratio $\nu = \rho_s/\rho_\ell$.

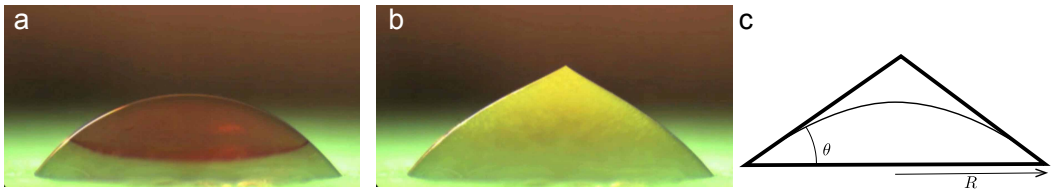


Figure 9: (a) Photograph of a partially frozen drop, after it is placed onto a cold surface. The bottom part of the drop has turned into solid ice (green), while the top part is still liquid (red). (b) Photograph of the fully solidified drop, exhibiting a conical tip. (c) The tip formation is modelled by a transformation from a liquid "spherical cap" to a solid "cone". Both the cap and the cone have the same base radius R and the same angle θ.

As a first model, we assume that the unfrozen liquid mass in Figure 9(a) can be approximated by a perfect spherical cap with a flat base, with base radius R and angle θ – see Figure 9(c) for a definition. The volume of a spherical cap reads $V_{\text{cap}} = R^3 f(\theta)$, with

$$f(\theta) = \frac{\pi}{3}\left(\frac{2 - 3\cos\theta + \cos^3\theta}{\sin^3\theta}\right). \tag{28}$$

The spherically-shaped liquid transforms into a cone of ice, with the same base radius R and the same angle θ (Figure 9(c)).

a) (1 pt) Show that the volume of the cone $V_{\text{cone}} = \frac{\pi}{3}\tan(\theta)R^3$.

(1 pt)
Answer 1: The volume of a pyramid or a cone reads $V = \frac{1}{2}Ah^2$, where A is the basal area and h the height. Using the geometric relations $A = \pi R^2$ and $h = R\tan\theta$, we find $V_{\text{cone}} = \frac{\pi}{3}\tan(\theta)R^3$.

Answer 2, for a slightly more complete derivation: An element of volume of

the cone reads $dV = Adh = \pi R^2 dh$. One verifies that $dh = \tan\theta dR$, so that $dV = \pi R^2 \tan\theta dR$. Integrating gives $V = \int_0^R d\tilde{R}\, \pi \tilde{R}^2 \tan\theta = \frac{\pi}{3}\tan(\theta) R^3$.

An equation for the angle θ is obtained by comparing the liquid mass stored in the spherical cap to the frozen mass of the final ice cone.

b) (2 pts) Introducing $x = \cos\theta$, show that mass conservation implies:

$$2x - 3x^2 + x^4 - \nu\left(1 - x^2\right)^2 = 0. \tag{29}$$

(2 pts) The mass inside the liquid cap reads $\rho_\ell V_{\text{cap}}$, which has to equal to final mass in the frozen cone $\rho_s V_{\text{cone}}$. Dividing both sides by ρ_ℓ gives the equation $f(\theta)R^3 = \nu\frac{\pi}{3}\tan(\theta)R^3$, or in explicit form

$$\frac{2 - 3\cos\theta + \cos^3\theta}{\sin^3\theta} = \nu\tan\theta. \tag{30}$$

Next, we introduce $x = \cos\theta$, so that $\sin^2\theta = (1 - x^2)$. Multiplying the equation by $\sin^3\theta$, this gives

$$2 - 3x + x^3 = \nu\frac{\sin^4\theta}{\cos\theta} = \nu\frac{(1-x^2)^2}{x}. \tag{31}$$

This can be rearranged to (29).

The physically relevant solution to this equation reads

$$x = \frac{1}{\sqrt{1-\nu}} - 1. \tag{32}$$

c) (2 pts) Determine the range of ν for which this model predicts a conical solution.

(2 pts) For a cone, the angle θ must lie in the range $0 < \theta < \pi/2$. This means that $x = \cos\theta$ must lie within $0 < x < 1$. The lower bound is reached for $\nu = 0$. The upper bound is reached when $1/\sqrt{1-\nu} = 2$, which is solved as $\nu = 3/4$. Hence, the formation of a cone requires $0 < \nu < 3/4$.

The model is able to explain the formation of a cone, but unfortunately it does not capture the phenomenon for the density ratio of water $\nu \approx 0.92$. The flaw of the model based on Figure 9(c) is the implicit assumption that the freezing front is perfectly flat. Figure 10 provides a more accurate view of the freezing front, which has two important geometric features: (i) the front has a spherical shape, (ii) the front ends perpendicularly to the ice-air interface.

d) (2 pts) Based on these refined assumptions, derive the mass conservation equation.

(2 pts) The liquid mass now consists of two spherical caps $\rho_\ell\left[f(\theta) + f(\pi/2 - \theta)\right]R^2$. This needs to be converted to the ice mass $\rho_s\left[f(\pi/2-\theta) + \frac{\pi}{3}\tan\theta\right]R^3$. The refined mass conservation equation thus reads

$$f(\pi/2 - \theta) + f(\theta) = \nu\left[f(\pi/2-\theta) - \frac{\pi}{3}\tan\theta\right]. \tag{33}$$

This could be worked out in further detail, but that is not necessary for answering the next question.

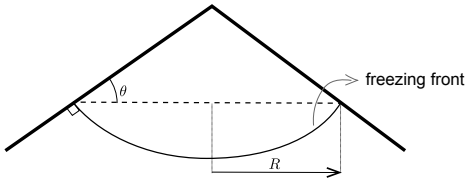


Figure 10: Sketch of the refined model. The freezing front has a (downward) spherical shape, that ends perpendicularly to the ice-air interface.

e) (1 pt) Using a graphical calculator, determine the angle θ for $\nu \approx 0.92$.

(1 pt) The solution to the refined model gives $\theta \approx 0.426$ rad, which corresponds to $\approx 24°$.

f) (2 pts) Provide a physical motivation why the freezing front is perpendicular to the ice-air interface.

(2 pts) Keywords: (i) latent heat during solidification. (ii) freezing front = isotherm (constant temperature). (iii) heat flux along temperature gradients, i.e. perpendicular to isotherms. (iv) negligible heat flux into the air, owing to the poor heat conductivity of air. (v). Combining these arguments, the isotherm of the freezing front must be perpendicular to the ice-air interface.

9 The relativity of electric and magnetic fields

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10 points

Consider a negative charge $q(-)$ moving with velocity $v_0 = v$ in frame **S** parallel to a current-carrying wire as in Figure 11. The density of the (negative) conduction electrons is ρ_- and their velocity is $v_- = v$. The density of the fixed, positive nuclear charges is ρ_+ with velocity $v_+ = 0$ in frame **S**. We consider an uncharged wire, so $\rho_+ = \rho_-$.

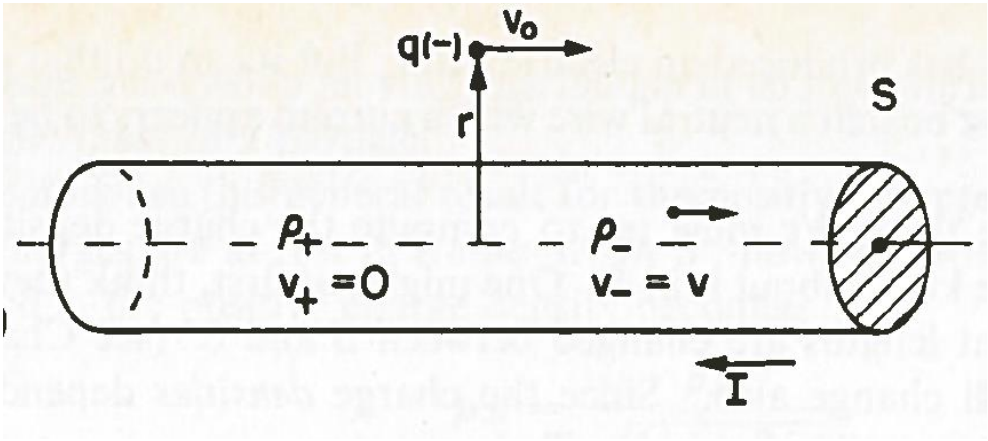


Figure 11: Charge $q(-)$ and current-carrying wire in laboratory frame **S** in which the wire is at rest.

a) (1 pt) What is the electric force F_{el} acting on charge $q(-)$ in frame **S**?

(1 pt) The electric force acting on a particle with a negative charge q is given by: $\vec{F} = -q\vec{E}$, however, the net charge density is neutral in frame **S**, which means that their is no electric field. Hence, the electric force is zero.

b) (1 pt) What is the magnetic force F_{mag} acting on charge $q(-)$ in frame **S**?

(1 pt)

$$F_{mag} = -q(\vec{v} \times \vec{B})$$
$$\vec{v} = v\hat{z}$$

To find magnetic field use the Ampere loop method:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\int_\Omega \vec{B} \cdot d\vec{l} = \int_S \mu_0 \vec{J} \cdot d\vec{A}$$

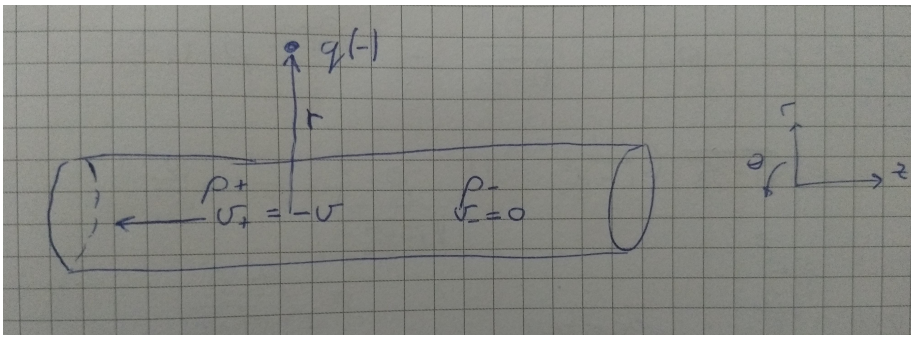
$$2\pi r B_\theta = -\rho_+ v \mu_0 A \qquad \text{where } A \text{ is the area of the cross section of the wire.}$$

$$\vec{B} = -\frac{\rho_+ v \mu_0 A}{2\pi} \frac{1}{r} \hat{\theta}$$

$$\Rightarrow F_{mag} = -\frac{q\rho_+ v^2 \mu_0 A}{2\pi} \frac{1}{r} \hat{r}$$

c) (1 pt) Draw a figure similar to Figure 11 for frame **S'** in which the charge $q(-)$ is at rest.

(1 pt)



d) (1 pt) What is the magnetic force F'_{mag} acting on charge $q(-)$ in frame **S'**?

(1 pt) In frame **S'** $v = 0$, so no magnetic force on particle.

e) (1 pt) Derive that the positive charge density ρ'_+ in frame **S'** is given by

$$\rho'_+ = \frac{\rho_+}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

(1 pt) In frame **S**:

$$\rho_+ = \frac{Q}{\Delta x} \qquad \text{where Q total charge in length } \Delta x$$

In frame **S'**:

$$\Delta x' = \frac{1}{\gamma} \Delta x = \sqrt{1 - \frac{v^2}{c^2}} \Delta x$$

$$Q' = Q,$$

due to length contraction. So:

$$\rho'_+ = \frac{\rho_+}{\sqrt{1 - \frac{v^2}{c^2}}}$$

f) (1 pt) Similarly, derive that the negative charge density ρ'_- in frame **S'** is given by

$$\rho'_- = \rho_- \sqrt{1 - \frac{v^2}{c^2}}.$$

(1 pt) In frame **S'**:

$$\rho'_- = \frac{Q'}{\Delta x'} \qquad \text{where Q' total charge in length } \Delta x'$$

In frame **S**:

$$\Delta x = \frac{1}{\gamma}\Delta x' = \sqrt{1 - \frac{v^2}{c^2}}\Delta x'$$
$$Q' = Q,$$

due to length contraction. So:

$$\rho'_- = \rho_- \sqrt{1 - \frac{v^2}{c^2}}$$

g) (1 pt) Give an expression for the total charge density ρ' in frame **S'** in terms of the positive charge density ρ_+ in frame **S**.

(1 pt)

$$\begin{aligned}\rho' &= \rho'_+ - \rho'_- \\ &= \rho_+ \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - \rho_+ \sqrt{1 - \frac{v^2}{c^2}} \\ &= \frac{\rho_+ v^2}{c^2 \sqrt{1 - \frac{v^2}{c^2}}}\end{aligned}$$

h) (1 pt) What is the electric force $F'_{\mathbf{el}}$ acting on charge $q(-)$ in frame **S'**?

(1 pt) The electric force on a particle is given by: $\vec{F} = -q\vec{E}$. To determine the electric field use a cylinder of length l and radius r as Gausbox:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$
$$\int_S \vec{E} \cdot d\vec{A} = \int_V \frac{\rho}{\epsilon_0} dV$$
$$= 2\pi r l E_r = \frac{\rho' A l}{\epsilon_0}$$
$$\vec{E} = \frac{\rho' A}{2\pi\epsilon_0} \frac{1}{r} \hat{r}$$
$$\Rightarrow F'_{el} = -\frac{q\rho' A}{2\pi\epsilon_0} \frac{1}{r} \hat{r}$$

i) (1 pt) Show that

$$F'_{\text{el}} = \frac{F_{\text{mag}}}{\sqrt{1-\frac{v^2}{c^2}}}.$$

(1 pt)

$$F'_{el} = -\frac{qA}{2\pi\epsilon_0}\frac{\rho_+ v^2}{c^2\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{r}\hat{r}$$

If we now use that $c^2 = \frac{1}{\mu_0\epsilon_0} \Rightarrow \mu_0 = \frac{1}{c^2\epsilon_0}$ we obtain the desired expression if we compare with the previous answer.

j) (1 pt) Explain that we obtain the same physical result in both frames.

(1 pt) The Lorentz transformation for a force perpendicular to the direction of movement is given by:

$$\vec{F'} = \gamma\vec{F},$$

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