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The uitwerkingen starten op de volgende pagina.

1. Train Problem

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11 punten

The two important equations in this problem are the continuity equation (conservation of air)

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0,$$

and the Euler equation (Newton's second law)

$$\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\frac{\nabla p}{\rho} + \vec{g}. \tag{1.1}$$

Furthermore, we assume that the equation of state is given by the ideal fluid formula

$$p = \rho R_{spec} T.$$

1.a) (1 pt) One should substitute $\vec{g}$ in (1.1) with $-\vec{a}$.

1.b) (2 pt) We assume the air to be in thermodynamic equilibrium. Hence $\partial_t \rho = 0$ and $\vec{v} = 0$. The continuity equation is now a trivial equation, while the Euler equation reduces to

$$-\frac{\nabla p}{\rho} - \vec{a} = 0.$$

Using the equation of state, one finds

$$\frac{\nabla \rho}{\rho} = -\frac{\vec{a}}{R_{spec} T}.$$

Solving this differential equation yields

$$\rho(x) = K e^{-ax/RT}.$$

1.c) (1 pt) The dimensionless density contrast is

$$\delta_{eq} = e^{-ax/RT} - 1.$$

Hence for a typical acceleration of 1m/s^2 (from 0 to 36 km/h in 10 sec), this implies that $\delta_{max} \simeq 1.15 \times 10^{-3} \ll 1$.

1.d) (2 pt) The equation for hydrostatic equilibrium is

$$\frac{dp}{dz} = -\rho g.$$

Water is an incompressible fluid (i.e. ρ = constant). Therefore, p is proportional to z. Hence, a human ear can feel a change in pressure of about $\frac{20 \text{ cm}}{10 \text{ m}} = 2\%$. Therefore an average person can feel a train accelerating in his ears if

$$\delta_{eq} = e^{-ax/RT} - 1 = \pm 0.02.$$

i.e. for $a \lesssim 16$ m/s^2, which is more than $1g$. A typical train is very unlikely to accelerate that fast, rather one estimates from 0 to 36 km/h in 10 sec, that $a \gtrsim 16$ m/s^2.

1.e) **(2 pt)** We substitute $\rho(x,t) = \bar{\rho}\,[1 + \delta(x,t)]$ in the continuity and the Euler equation and consider $\delta(x,t)$, v and their derivatives to be small. This yields up to first order

$$\partial_t \delta + \bar{\rho}\nabla v = 0,$$

$$\partial_t v = -\frac{c_s^2}{\bar{\rho}}\nabla\delta.$$

Taking the gradient of the second equation, the time derivative of the first equation and substituting the second into the first equation yields

$$\ddot{\delta} - c_s^2\nabla^2\delta = 0.$$

The solutions are good old waves $\delta(x,t) = f_+(x - c_s t) + f_(x + c_s t)$, with speed of sound c_s. The frequency ω (related to the wavenumber k by $\omega = c_s k$) depends on the initial conditions.

1.f) **(3 pt)** One finds using that $c_s^2 = \frac{dp}{d\rho} = RT$ up to first order

$$\dot{\delta} + \nabla v - \frac{av}{RT} = 0, \qquad \dot{v} = -c_s^2\nabla\delta.$$

Taking the time derivative of the first and using the second equations (and its gradient), one finds

$$\ddot{\delta} = c_s^2\left[\nabla^2\delta - \frac{a}{RT}\nabla\delta\right].$$

Separation of variables gives the two ordinary differential equations

$$\ddot{f} = \omega^2 f, \quad g'' - \frac{a}{RT}g' + \left(\frac{\omega}{c_s}\right)^2 g = 0,$$

with solutions

$$f(t) = \cosh\left[\omega(t - t_0)\right],$$

$$g(x) = \exp(\frac{ax}{2RT})\left[A\cos\left(x\sqrt{\left(\frac{\omega}{c_s}\right)^2 - \left(\frac{a}{2RT}\right)^2} + B\right)\right],$$

with t_0, A and B integration constants. The boundary conditions require $v(-L/2) = v(L/2) = 0$. Since v is proportional to g', from this one can solve for the allowed ω and the relative $k = \omega/c_s$.

2. Non-interacting spins

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2.a) (1 pt) The partition function of a single spin is

$$Q_1(H,T) = \sum_{S_i \in \{+1,-1\}} e^{-\beta \mathcal{H}(S_i)} = e^{+\beta H} + e^{-\beta H} = 2\cosh(\beta H).$$

And the probabilities p_+ and p_- are

$$p_+ = \frac{e^{-\beta\mathcal{H}(+1)}}{Q_1} = \frac{e^{+\beta H}}{e^{+\beta H} + e^{-\beta H}}$$
$$p_- = \frac{e^{-\beta\mathcal{H}(-1)}}{Q_1} = \frac{e^{-\beta H}}{e^{+\beta H} + e^{-\beta H}}$$

2.b) (2 pt) The average number of up spins $\langle N_+ \rangle$ is just the probability of being up p_+ times the amount of particles N. The same process is valid for $\langle N_- \rangle$.

$$\langle N_+ \rangle = \frac{Ne^{+\beta H}}{e^{+\beta H} + e^{-\beta H}}$$
$$\langle N_- \rangle = \frac{Ne^{-\beta H}}{e^{+\beta H} + e^{-\beta H}}$$

And the average energy can be calculated as follows

$$\langle E \rangle = N_+E_+ + N_-E_- = -NH\left(\frac{e^{+\beta H} - e^{-\beta H}}{e^{+\beta H} + e^{-\beta H}}\right) = -NH\tanh(\beta H).$$

2.c) (2 pt) The partition function for the total system in terms of the total amount of spins N and the amount of up spins N_+ is

$$Q(N, N_+, H, T) = \frac{N!}{N_+!(N-N_+)!} e^{-\beta H[-N_+ + (N-N_+)]},$$
$$= \frac{N!}{N_+!(N-N_+)!} e^{+\beta H[2N_+ - N]}.$$

Starting from $F = -k_BT\log Q$, it follows that

$$F = -k_BT\left[\log\frac{N!}{N_+!(N-N_+)!} + \beta H(2N_+ - N)\right],$$
$$= -k_BT\left[\log N! - \log N_+! - \log(N-N_+)! + \beta H(2N_+ - N)\right].$$

Using the Stirling approximation ($\log N! \approx N\log N - N$).

$$\begin{aligned}\frac{F}{k_BT} &= -\log N! + \log N_+! + \log(N-N_+)! - \beta H(2N_+ - N),\\ &\approx -N\log N + N + N_+\log N_+ - N_+ \; + \\ &\quad (N-N_+)\log(N-N_+) - (N-N_+) + \beta H(N-2N_+),\end{aligned}$$

All the loose N and N_+ terms cancel each other out. Next we add 0 in the form of $(N_+ - N_+)\log N$ to our equation and put those as denominators in the log terms with the same pre-factor.

$$\begin{aligned}\frac{F}{k_BT} &= -(N - N_+ + N_+)\log N + N_+\log N_+ + (N-N_+)\log(N-N_+) + \beta H(N-2N_+),\\ &= N_+\log\frac{N_+}{N} + (N-N_+)\log\frac{N-N_+}{N} + \beta H(N-2N_+).\end{aligned}$$

Which is the desired result.

2.d) (2 pt) We have

$$S = -\left(\frac{\partial F}{\partial T}\right)$$

The factor T just vanishes in the expression and we pick up a minus sign. Also the last term disappears since, when both sides are multiplied with $k_BT = 1/\beta$, it is no longer dependent on the temperature T.

$$S = -k_B\left[N_+\log\frac{N_+}{N} + (N-N_+)\log\frac{N-N_+}{N}\right].$$

If $m = p_+ - p_-$, and we use that $N_\pm/N = p_\pm$, then we can just rewrite our expression into

$$\frac{S}{N} = -k_B\left[p_+\log p_+ + p_-\log p_-\right].$$

And when we now use $p_+ + p_- = 1$ to express $2p_\pm = 1 \pm m$ our expression turns into the desired result

$$\frac{S}{N} = -k_B\left[\frac{1+m}{2}\log\frac{1+m}{2} + \frac{1-m}{2}\log\frac{1-m}{2}\right].$$

2.e) (1 pt) We can use the same definitions of the probabilities $2p_\pm = 1 \pm m$ to express our answer from b) into m.

$$\langle E\rangle = p_+E_+ + p_-E_- = \frac{1+m}{2}(-H) + \frac{1-m}{2}(H) = -mNH.$$

2.f) (2 pt) We can use here that the temperature is defined through

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right) = \left(\frac{\partial S}{\partial m}\right)\left(\frac{\partial m}{\partial E}\right),$$
$$= -Nk_B\left[\frac{1}{2}\log\frac{1+m}{2} + \frac{1+m}{2}\frac{2}{1+m} - \frac{1}{2}\log\frac{1-m}{2} + \frac{1-m}{2}\frac{2}{1-m}(-1)\right]\left(\frac{\partial m}{\partial E}\right),$$
$$= \frac{-Nk_B}{2}\log\left[\frac{1+m}{1-m}\right]\left(-\frac{1}{NH}\right),$$
$$\frac{1}{T} = \frac{k_B}{2H}\log\left[\frac{1+m}{1-m}\right].$$

Now when we have $m < 0$, then $(1-m) > (1+m)$ so then $\log\left[\frac{1+m}{1-m}\right]$ returns a negative value for the temperature T. Note that we implicitly assumed $H > 0$.

2.g) (2 pt) For system 1 with $T_1 < 0$ we consider a change in entropy due to an infinitesimal decrease in energy equal to $\delta > 0$.

$$\Delta S_1 = S_1(E-\delta) - S_1(E),$$
$$= \frac{S_1(E-\delta) - S_1(E)}{\delta}\delta.$$

for small δ this becomes

$$\Delta S_1 = -\left(\frac{\partial S_1}{\partial E}\right)\delta = -\frac{\delta}{T_1} \geq 0.$$

The calculation for ΔS_2 is similar and also yields a positive change in entropy. So, you gain entropy by donating some energy from a system with negative temperature to a neighbouring system with positive temperature.

3. Bound states at edges

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We consider a specific tight binding model of electrons hopping on a square lattice, where on each lattice site there are two orbital states. Assuming periodic boundary conditions in two directions the Bloch Hamiltonian can be written as

$$H(k_x, k_y) = \begin{pmatrix} -2 - \cos k_x - \cos k_y + m & -\sin k_x + i \sin k_y \\ -\sin k_x - i \sin k_y & 2 + \cos k_x + \cos k_y - m \end{pmatrix}$$

3.a) (2 pt) The dispersion relation is the energy as a function of momentum, hence the eigenvalues of $H(k_x, k_y)$ as a function of k_x and k_y. They are

$$E_\pm(k_x, k_y) = \pm\sqrt{2\cos(k_x)(\cos(k_y) - m + 2) - 2(m-2)\cos(k_y) + (m-4)m + 6}$$

3.b) (1 pt) We Taylor expand cos and sin around $\pm\pi$ i.e. we write

$$\cos(\pi + \Delta k) \approx -1 \text{ and}$$
$$\sin(\pi + \Delta k) \approx -\Delta k.$$

This yields

$$H(\pi + k_x, \pi + k_y) \approx \begin{pmatrix} m & k_x - ik_y \\ k_x + ik_y & -m \end{pmatrix}.$$

One may verify that this agrees with the given real space Bloch Hamiltonian, by applying it to the two-component spinor given by

$$\psi(x, y) = \begin{pmatrix} e^{(ik_x x + ik_y y)/v_F \hbar} \\ e^{(ik_x x + ik_y y)/v_F \hbar} \end{pmatrix}.$$

3.c) (1 pt) Using the approximation for the cosine again, we obtain

$$\begin{aligned} E_+(k_x, k_y) &\approx \sqrt{2(m-1) + 2(m-2) + m(m-4) + 6} \\ &= \sqrt{4m - 6 + m^2 - 4m + 6} \\ &= |m|. \end{aligned}$$

3.d) (2 pt) Here one may use the ansatz

$$\psi(x, y) = \psi(x) e^{ik_y},$$

where $\psi(x)$ is an arbitrary two-component spinor that only depends on x. This ansatz basically tells us that we may replace ∂_y with k_y, which yields the desired result.

3.e) (3 pt) We are looking for a zero energy state, hence we must solve the equation

$$H(k_y, x)\vec{\phi}e^{\lambda(x)}0.$$

Substitution of the expression for $H(k_y, x)$ at $k_y = 0$ leads to the following system of effective equations for λ

$$m(x)\phi_1 - iv_F\hbar\partial_x\lambda(x)\phi_2 = 0,$$
$$m(x)\phi_2 + iv_F\hbar\partial_x\lambda(x)\phi_1 = 0.$$

The solution is found by integrating

$$\lambda(x) = \frac{-i}{v_F\hbar}\frac{\phi_1}{\phi_2}\int_0^x dx\; m(x),$$

$$\lambda(x) = \frac{i}{v_F\hbar}\frac{\phi_2}{\phi_1}\int_0^x dx\; m(x).$$

This equation can only be consistent if

$$\frac{\phi_1}{\phi_2} = -\frac{\phi_2}{\phi_1}.$$

These are two solutions $\phi_1/\phi_2 = \pm i$, leading to two solutions for the wave function

$$\psi_\pm(x, k_y = 0) = \vec{\phi}\; e^{\frac{\pm 1}{v_F\hbar}\int_0^x dx\; m(x)}.$$

From the given mass profile $m(x)$, it follows that $\int_0^x dx\; m(x)$ is a decreasing function of x, hence ψ_+ is the normalizable solution. This solution is exponentially localized around $x = 0$.

3.f) (2 pt) We first rewrite the Hamiltonian for non-zero k_y as

$$H(k_y, x) = H(0, x) + \begin{pmatrix} 0 & -iv_F\hbar k_y \\ iv_F\hbar k_y & 0 \end{pmatrix}.$$

The first part annihilates the normalizable wave-function ψ_+, as shown in part (e). We may now compute

$$H(k_y, x)\psi_+(x, k_y) = \begin{pmatrix} 0 & -iv_F\hbar k_y \\ iv_F\hbar k_y & 0 \end{pmatrix}\vec{\phi}e^{\frac{1}{v_F\hbar}\int_0^x dx\, m(x)}.$$

We recall from part (e) that the solution ψ_+ corresponds to $\phi_1/\phi_2 = i$, hence we may pick $\phi_1 = i$ and $\phi_2 = 1$, (this will not be normalized, but this does not change the argument). Using this, one may show that

$$H(k_y, x)\psi_+(x, k_y) = -v_F\hbar k_y\psi_+(x, k_y).$$

The group velocity in the y-direction is given by

$$\frac{\partial H}{\partial k_y} = -v_F\hbar,$$

thus there is a current running in the negative y-direction along the boundary (x = 0).

4. Low energy photons and gauge invariance

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4.a) (2 pt) Using conservation of momentum $p' = p + k$ and the on-shell condition for $p^2 = m^2$,

$$M\left(\phi(p')\right) \to \phi(p)\gamma(k)) = \frac{q(2p+k)\cdot\epsilon^*(k)}{(p+k)^2 - m^2} = \frac{qp\cdot e^*_\mu(k)}{p\cdot k}. \tag{4.1}$$

4.b) (2 pt) Iterating and summing over the ordering in which the photons are emitted

$$\begin{aligned} M(\phi(p') \to \phi(p)\gamma(k_1)...\gamma(k_n)) &= \sum_{order} \frac{q(2p+k_1+...+k_n)\cdot\epsilon^*(k_1)}{(p+k_1+...k_n)^2 - m^2} \times ... \times \frac{q(2p+k_n)\cdot\epsilon^*(k_n)}{(p+k_n)^2 - m^2} \\ &= \sum_{order} \frac{qp\cdot\epsilon^*(k_1)}{p\cdot(k_1+...+k_n)} \times ... \times \frac{qp\cdot\epsilon^*(k_n)}{p\cdot k_n} \\ &= \frac{qp\cdot\epsilon^*(k_1)}{p\cdot k_1} \times ... \times \frac{qp\cdot\epsilon^*(k_n)}{p\cdot k_n} \end{aligned}$$

Directly writing down the last line is not correct. Rather than summing over the ordering one may also argue this result by taking $p\cdot k_1 \ll ... \ll p\cdot k_n$, in which case other orderings are suppressed.

4.c) (2 pt) For a single photon

$$\langle\gamma(k)|iq\int_0^\infty ds \int \frac{d^4k'}{(2\pi)^4} e^{-isp\cdot k'} p\cdot A(k')|0\rangle = q\int \frac{d^4k'}{(2\pi)^4}\frac{p_\mu}{p\cdot k'}\langle\gamma(k)|A^\mu(k')|0\rangle = \frac{qp\cdot\epsilon^*(k)}{p\cdot k}$$

This directly generalizes to n photons. There, the $1/n!$ from the exponential cancels against the $n!$ ways the fields produces the photons.

4.d) (2 pt) From the form of the equation for the production of low-energetic photons, it is clear that they are only sensitive to the momentum p and electric charge q. In particular, the expression is the same, independent of the spin of the emitting particle.

4.e) (2 pt)

$$\delta_\xi\left(\int_0^\infty ds\ p\cdot A(sp^\mu)\right) = \int_0^\infty ds\ p\cdot\partial\xi(sp^\mu) = \int_0^\infty ds\ \frac{d}{ds}\xi(sp^\mu) = -\xi(0)$$

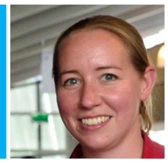
so it leads to a factor of $e^{-iq\xi(0)}$.

4.f) (2 pt) We get a similar phase for each particle in the initial and final state, but with the opposite phase for incoming particles as can be argued from equation (4.1). This leads to gauge invariance

$$\prod_{i\in in} ie^{iq_i\xi(0)} \prod_{i\in out} ie^{-iq_i\xi(0)} = \exp\Big[i\Big(\sum_{i\in in} q_i - \sum_{i\in out} q_i\Big)\xi(0) = 1 \ .$$

Due to conservation of electric charge

$$\sum_{i\in in} q_i = \sum_{i\in out} q_i$$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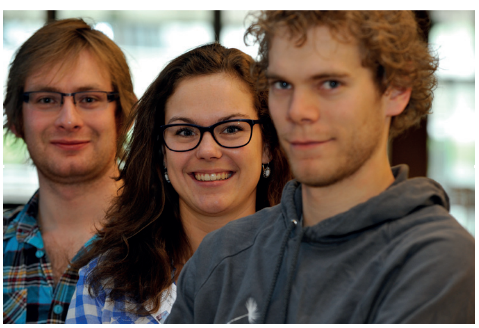


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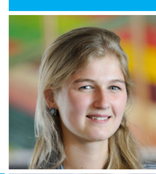
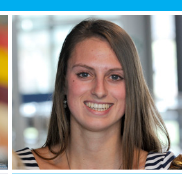

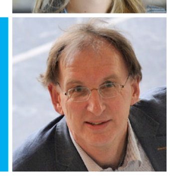


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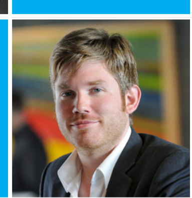

5. Shockley-Queisser limiet

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5.a) (1 pt) Dit betekent dat we alleen het hogere frequentiegebied kunnen benutten. De totaal verloren energie is dus

$$E_{lost} = \Delta t \Omega A \int_0^{\frac{E_g}{h}} \frac{2h\nu^3}{c^2(e^{\frac{h\nu}{k_B T}} - 1)} d\nu$$

waar Δt de totale verstreken tijd is, Ω de ruimtehoek van het zwarte lichaam, en A het frontale oppervlak van het zonnepaneel.

5.b) (1 pt) Het aantal fotonen van een bepaalde frequentie wordt gegeven door

$$N_{ph} = \frac{h\nu^2}{c^2(e^{\frac{h\nu}{k_B T}} - 1)} d\nu$$

Deze verliezen ieder een energie $h\nu - E_g$. Dus de totaal in dit kanaal verloren energie is.

$$E_{lost} = \Delta t \Omega A \int_{\frac{E_g}{h}}^{\infty} \frac{\nu^2(h\nu - E_g)}{c^2(e^{\frac{h\nu}{k_B T}} - 1)} d\nu$$

5.c) (2 pt) Als we de antwoorden voor vragen a) en b) combineren (of ons realiseren dat er van alle gebruikte fotonen een energie E_g nuttig gebruikt wordt), vinden we de volgende formule voor de totale elektrische energie

$$E_{el} = \Delta t \Omega A E_g \int_{\frac{E_g}{h}}^{\infty} \frac{\nu^2}{c^2(e^{\frac{h\nu}{k_B T}} - 1)} d\nu$$

Het gebruik van de benadering van Wien en tweemaal partieel integreren geeft

$$E_{el} = \Delta t \Omega A E_g \frac{k_B T}{hc^2} \left(2(\frac{k_B T}{c^2})^2 + 2\frac{k_B T}{h}\frac{E_g}{h} + (\frac{E_g}{h})^2 \right) e^{-\frac{E_g}{k_B T}}$$

$$E_{total} = 6\Delta t \Omega A \int_0^{\infty} \frac{2h\nu^3}{c^2 e^{\frac{h\nu}{k_B T}}} d\nu = \Delta t \Omega A E_g \frac{(k_B T)^4}{h^4 c^2}$$

$$\eta = \frac{E_{el}}{E_{total}} = \frac{E_g h^2}{6(k_B T)^3} \left(2(\frac{k_B T}{c^2})^2 + 2\frac{k_B T}{h}\frac{E_g}{h} + (\frac{E_g}{h})^2 \right) e^{-\frac{E_g}{k_B T}}$$

5.d) (2 pt) Als we $R = E_g/k_BT$ definiëren, reduceert dit tot

$$\eta = \frac{1}{6}R(R + 2R + R^2)e^{-R}$$

5.e) (2 pt) Voor kleine bandgaps zal geen hoog rendement gehaald kunnen worden, omdat alle energie in de relaxatie gaat zitten, per foton kan maar een te kleine hoeveelheid energie nuttig gebruikt worden. Een grotere bandgap geeft per foton weliswaar meer energie, maar naar mate de bandgap groter wordt, zal het aantal fotonen dat genoeg energie heeft te klein worden. Hiertussen moet dus wel een optimum liggen.

5.f) (2 pt) Mogelijk ideeën hier zijn:

- Het plaatsen van twee halfgeleiders achter elkaar, eerst een met een hoge bandgap en daarna een met een lagere om de niet opgevangen fotonen ook nog om te zetten
- Opconversie van fotonen met een lage energie naar fotonen met hogere energie door nanodeeltjes in de zonnecel.
- Hete elektron vangst, het invangen van de meest energetische fotonen om daar los de energie van te extraheren.
- Fluorescente neerconversie, het coaten van de zonnecel met een materiaal dat hoog-energetische fotonen absorbeert en deze als twee laag-energetische uitzend.
- Meervoudige exiton generatie. Zorgen dat de zonnecel uit een foton meerdere excitonen kan maken.

5.g) (2 pt) Het vinden van het maximum geeft

$$R = \frac{1}{3}\left(1 + \sqrt[3]{37 - 3\sqrt{114}} + \sqrt[3]{37 + 3\sqrt{114}}\right) \approx 2.27$$

Dit invullen geeft $\eta_{max} = 0.457$ voor $E_G = 1.17eV$. Dit komt zeer goed overeen met de band gap van silicium, $1.12eV$.

6. Relativistische velden

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6.a) (2 pt) Dit is te zien door een Taylor-benadering van $U(a)$ toe te passen op $\varphi(x)$, en de resulterende uitdrukking te herkennen als de Taylor-benadering van $\varphi(x+a)$.

6.b) (1 pt) De vereenvoudiging leidt tot

$$U(L,0)^{-1}U(a)U(L,0) = U(L^{-1}a).$$

6.c) (2 pt) Dit volgt uit de berekening

$$\begin{aligned} U(a)U(L,0)\varphi(x) &= U(L,0)U(L,0)^{-1}U(a)U(L,0)\varphi(x) \\ &= U(L,0)U(L^{-1}a)\varphi(x) \\ &= e^{ip\cdot L^{-1}a}U(L,0)\varphi(x), \end{aligned}$$

Dus we lezen af: $p' = (p^T L^{-1})^T = Lp$ (tweede gelijkheid: Lorentz-transformaties zijn orthogonaal t.o.v. het Minkowski inproduct.)

6.d) (1 pt) Uit het antwoord van (c) en uit het feit dat Lorentz-transformaties orthogonaal zijn ten opzichte van het Minkowski inproduct volgt: $p' \cdot p' = m^2$.

6.e) (2 pt) Uit de kettingregel voor differentiëren volgt

$$\begin{aligned} \partial_\mu \partial^\mu \varphi(Lx+a) &= L^\mu_\rho \partial_\mu \partial^\rho \varphi(y)|_{y=Lx+a} \\ &= L^\mu_\rho L^\sigma_\mu \partial_\sigma \partial^\rho \varphi(y)|_{y=Lx+a} \\ &= \partial_\sigma \partial^\sigma \varphi(y)|_{y=Lx+a} \\ &= -m^2 \varphi(y)|_{y=Lx+a} \\ &= -m^2 \varphi(Lx+a). \end{aligned}$$

6.f) (2 pt) Eerst berekenen we het product,

$$\begin{aligned} &\left(-i\partial_t + i\alpha_i \partial^i - \beta m\right)\left(i\partial_t + i\alpha_j \partial^j - \beta m\right) \\ &= \partial_t^2 - \alpha_i \alpha_j \partial^i \partial^j - im\alpha_i \beta \partial^i - im\beta\alpha_i \partial^i + \beta^2 m^2 \\ &= \partial_t^2 - \frac{\alpha_i \alpha_j + \alpha_j \alpha_i}{2} \partial^i \partial^j - im\left(\alpha_i \beta + \beta \alpha_i\right)\partial^i + \beta^2 m^2 \end{aligned}$$

In aanvulling op de gegeven eis is dus verder nodig dat

$$\beta^2 = \mathrm{Id} \quad \text{en dat} \quad \alpha_i \beta + \beta \alpha_i = 0,$$

in dit geval impliceert de Dirac-vergelijking de Klein-Gordon vergelijking.

6.g) (2 pt) Hier volstaan de Pauli matrices. Echter zoals gezegd in de opgave, zijn er meerdere goede antwoorden. (Namelijk, als $\alpha_1, \alpha_2, \alpha_3$ voldoen aan

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \mathrm{Id}_{2\times 2},$$

en U is een inverteerbare matrix, dan voldoen $U\alpha_i U^{-1}$ ook aan bovenstaande vergelijking.)

7. The Amsterdam problem

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8 punten

7.a) (8 pt) Newton's second law implies for this static situation

$$\sum_i F_i = 0. \tag{7.1}$$

In this problem there are four forces working on the rope:

- The force of the mass attached to the rope, which is $F_M = Mg$
- The force, F, of the person pulling the rope
- The normal force, F_N, of the pulley on the rope
- The friction force, which is $F_f = \mu F_N$

The last two forces are locally defined, therefore we consider the problem for a small angle $\Delta\phi$. For this small angle we can write down equation (7.1) in the x and y component:

$$F\cos\left(\frac{\Delta\phi}{2}\right) = F_M\cos\left(\frac{\Delta\phi}{2}\right) + F_f$$
$$F_N = F\sin\left(\frac{\Delta\phi}{2}\right) + F_M\sin\left(\frac{\Delta\phi}{2}\right)$$

These equations reduce up to first order in $\Delta\phi$ to

$$F = F_M + \mu F_N,$$
$$F_N = F\frac{\Delta\phi}{2} + F_M\frac{\Delta\phi}{2}.$$

Hence,

$$F - F_M = \frac{\mu\Delta\phi}{2}(F + F_M).$$

Now, write $F_M = F + \Delta F$ and notice that $\Delta F = O(\Delta\phi)$, then

$$\Delta F = -\mu F\Delta\phi,$$

which reduces in the limit $\Delta\phi \to 0$ to the differential equation

$$\frac{dF}{d\phi} = -\mu F$$

that can be solved by separation of variables yielding

$$F(\phi) = C_0 e^{-\mu\phi}.$$

Using that $F(0) = F_M = Mg$ and $F_{min} = F(\varphi)$ one finds

$$F_{min} = Mge^{-\mu\varphi}.$$

8. Gravitational effects of sea level change

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12 punten

8.a) (2 pt) Mass is taken from the oceans of Earth to form the ice sheet, so the new potential becomes

$$\phi^*(r,\theta) = \frac{G(M_e - M_i)}{r} + \frac{GM_i}{x},$$

hence

$$\phi^*(R,\theta) = \frac{G(M_e - M_i)}{R} + \frac{GM_i}{2R\sin(\theta/2)}.$$

8.b) (2 pt) Now we want to taylor the expression found in a)

$$\phi^*(R+\epsilon,\theta) = \phi^*(R,\theta) + \epsilon \left.\frac{\partial\phi^*(r,\theta)}{\partial r}\right|_{r=R} + \mathcal{O}(\epsilon^2),$$

which simplifies to

$$\phi^*(R+\epsilon,\theta) \approx \phi^*(R,\theta) - \epsilon g.$$

(since the gravitational acceleration g is the derivative of the gravitational potential)

$$\phi(R,\theta) \approx \phi^*(R,\theta) - \epsilon g.$$

The equation for $\phi^*(R+\epsilon,\theta)$ is an equipotential surface and equals the new equipotential at the surface of the earth ϕ(R,θ).

8.c) (2 pt) We start by equating these potential surfaces to find an expression for ϵ.

$$\frac{GM_e}{R} = \frac{G(M_e - M_i)}{R} + \frac{GM_i}{2R\sin(\theta/2)} - \epsilon g,$$

$$\epsilon g = \frac{GM_i}{R}\left(\frac{1}{2R\sin(\theta/2)} - 1\right),$$

$$\epsilon = \frac{RM_i}{M_e}\left(\frac{1}{2R\sin(\theta/2)} - 1\right).$$

8.d) (2 pt) ($S_e = \Delta S_e$)

$$\epsilon = \frac{RM_i}{M_e}\left(\frac{1}{2R\sin(\theta/2)} - 1\right),$$

$$4\pi R^2 S_e + 2\pi R^2 \int_0^\pi \epsilon \sin\theta d\theta = -\frac{M_i}{\rho_w},$$

$$\int_0^\pi \left(\frac{1}{2R\sin(\theta/2)} - 1\right)\sin\theta d\theta = 0$$

The whole integral is zero, so we can now find the expression for S_e

$$S_e = -\frac{M_i}{4\pi R^2\rho_w} = -\frac{M_i}{4\pi R^2\rho_w}\frac{M_e}{M_e},$$
$$= -\frac{M_i}{4\pi R^2\rho_w}\frac{4\pi R^3\rho_e}{3M_e},$$
$$= -\frac{RM_i}{M_e}\frac{\rho_e}{3\rho_w}.$$

Finally

$$\epsilon^*(\theta) = \epsilon + S_e = \frac{RM_i}{M_e}\left(\frac{1}{2\sin(\theta/2)} - 1 - \frac{\rho_e}{3\rho_w}\right)$$

8.e) (2 pt) The global mean change is

$$S_e = -\frac{RM_i}{M_e}\frac{\rho_e}{3\rho_w},$$

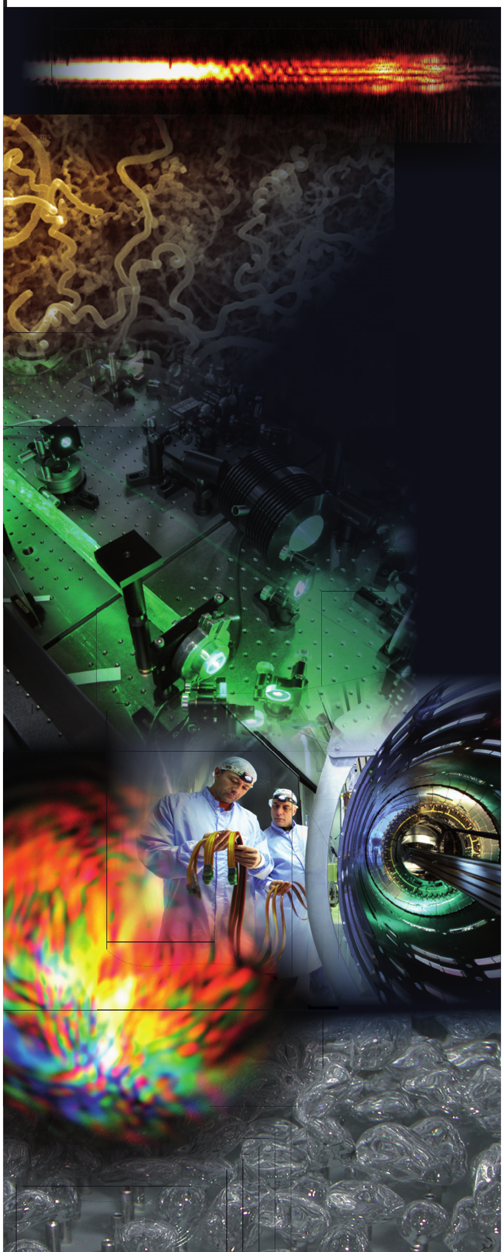
so

$$\frac{\epsilon^*(\theta)}{S_e} = \frac{\epsilon + S_e}{S_e} = -\frac{3\rho_w}{\rho_e}\left(\frac{1}{2\sin(\theta/2)} - 1 - \frac{\rho_e}{3\rho_w}\right).$$

The reduction in gravitational pull between ice and ocean as a consequence of the reduced ice mass will lead to a sea level lowering near the ice margin which can be seen in the equation above if you fill in $\theta = 5$, ϵ^* will be negative. Hence you will not get wet feet as the sea level will drop nearby the ice sheet.

8.f) (2 pt) M_i drops out if you calculate the ratio, so you just have to estimate θ (distance from G to NL (θ=25°) and A (θ>180°) to NL expressed in degrees and find the other constants from standard tables on this (ρ_e=5500 and ρ_w=1000). If you do this correctly you find approximately a factor 4.5 for A/G at the Dutch coast. (G to NL is a little ambiguous depending on where the melts takes place so answers between 20 and 30 for θ are ok). The implication is that it matters a lot whether the ice mass loss is from Greenland or Antarctica if the amount is similar. In fact mass loss from Greenland is not very important for NL, whereas mass loss from Antarctica is very important.

Einde van de uitwerkingen.



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