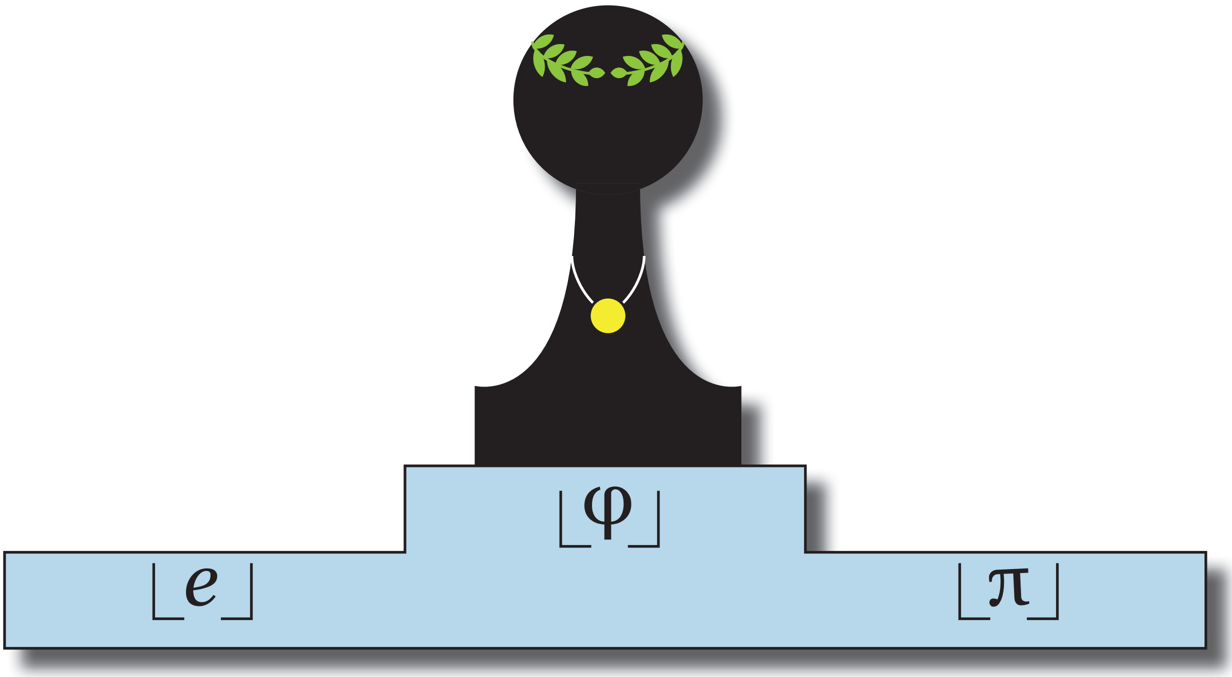
ANSWERS
PION 2014



Are you the chamPIONs in physics?

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Antwoorden PION 2014 -

Bycicle Physics

(a) Arbeid (en vermogen) is kracht maal weg (kracht maal weg per tijdseenheid). Als je snel gaat is de weg (per seconde groot) en de kracht die je moet compenseren dus klein. De tegenwerkende kracht is bij 40 km/uur dus in absolute kleiner dan bij 15 km/uur helling op. De aandrijving op een fiets is niet constant. Om twee redenen: gedurende een omwenteling van de trapper is de kracht niet constant. Als de trapper boven of onder is duw je wel op de trapas maar heb je geen aandrijving, als de trapper horizontaal staat is er maximale krachtmoment). De tweede reden is dat je als mens niet makkelijk een constant vermogen kan leveren, er zijn natuurlijke langzame variaties.

Als tijdens de omwenteling van de trapper de aandrijfkracht kleiner wordt (of even nul wordt) zal de fiets op de helling een veel grotere absoute vertraging tot gevolg hebben dan op het vlakke, vanwege *F=ma* (met *a* negatief, de vertraging) en *F* de externe krachten op de fiets. De fietser voelt dit als een voortdurende verandering van de snelheid op een helling die bijna afwezig is op het vlakke.

Realisatie: Realisatie dat tegenwerkende kracht in steady state veel groter is bij lage snelheid: 2 punt (eventueel met invullen van getallen in een F=ma formule)
Realisatie dat fluctuaties gedurende de omwenteling of natuurlijk daarom een heel verschillend gevoel geven : 1 punt

(b) Bij een aandrijfmechanimse dat niet cyclisch is, maar een constant vermogen vraagt en dat ook levert, zou het verschil veel kleiner zijn. Bij een constante aandrijving blijft de snelheid dan simpelweg constant en heb je geen last meer van de fluctuaties door de circulaire beweging alleen nog van de vermoeidheidsfluctuaties. Overigens was het antwoord van Lance Armstroing om zichzelf een heel hog ritme (>120 /minuut) aan te trainen op de hellingen in tegenstelling tot veel fietsers die juist naar een lager ritme gaan)
Antwoord circuaire trapper beweging omzetten in iets constant: 1.5 punt

(c) Om een fiets (+ berijder) te versnellen moet er een externe kracht zijn. Dit is de kracht van de weg op de fiets opgeroepen als reactiekracht door de fietser op het achterwiel.
De weg oefent krachten uit op beide wielen: op het achterwiel naar voren (in dezelfde richting dat je versnelt en in dezelfde richting dat de ketting trekt aan de kransjes op het achterwiel), maar zodanig dat het netto krachtmoment het wiel de goede kant laat draaien. (Dus het moment van de ketting op de kransjes: $F_{ketting}$ x afstand krasnje tot as is groter dan de kracht van de weg op het achterwiel F_{weg} x de straal van het wiel. Het kleine verschil laat het wiel in de goede richting sneller draaien. De absolute kracht F_{weg} zorgt voor versnelling van het geheel)
Echet om het voorwiel de goed kant te laten versnellen moet er een kracht tegen de richting van de beweging in op het wiel uitgeoefend worden (er is een moment nodig voor de hoekversnelling van het voorwiel). De weg oefent dus een kracht uit op het voorwiel in de richting tegen de voortbeweging. Met andere woorden: de totale kracht van de weg op het achterwiel *F* > ma om de kracht in tegengestelde richting op het voorwiel te compenseren.

Opschrijven dat de weg een kracht geeft op het achteriwel die voor versnelling zorgt:
Analyse van de grootte van de kracht met krachtmoment ketting en krachtmoment weg en netto hoekversnelling en lineaire versneling: 1 punt

Beschrijving krachten op voorwiel: in tegengestelde richting: 1 punt

(d) Als de gravitatie dominant is, levert wet van behoud van energie het snelle antwoord:
Dus: $\frac{1}{2}mv^2 = mgLsin(\alpha)$. Dit geeft: $L = \frac{v^2}{2gsin(\alpha)} = 0.82$ meter (waarbij we de 0.1 m/s maar verwaarlozen)
De andere is lastiger: eerste de effectieve kracht: vermogen bij klimmen is verandering in mgh dus uit mgv$_{verticaal}$: ofwel $mgvsin(\alpha) = 388$ Watt. Bij 43,2 km/uur = 12 m/s betekent dit dat de windkracht die tegenwerkt gelijk is aan: $F = \frac{388}{12} = 32.3\ N$. Deze kracht schrijven we als F_{wind}=cv^2waaruit volgt: *c=0.2245 Ns^2/m^2* is de coefficient van windkracht. Dezekracht is snelheidsafhankelijk en dus wordt de vertraging alsmaar kleiner.
De differentiaalvergelijking die de snelheidsverandering in de tijd geeft is:
$m\frac{dv}{dt} = -cv^2$ met als oplossing: $v(t) = \frac{v_0}{\frac{cv_0}{m}t+1}$. De constante v_0 is de beginsnelheid. De randvoorwaarde is dat omvallen gebeurt bij 0.1 m/s.
Oplossen voor de tijd dat de snelheid gelijk wordt aan de eindsnelheid, v_e geeft $t_e = \frac{m}{cv_0}\left(\frac{v_0}{v_e} - 1\right)$, *$t_{eind}$=3533* sec. Integreren van v(t) levert de verplaatsing

$$x(t) = \frac{m}{c}\ln\left(\frac{cv_0}{v_e}(t+1)\right)$$

En dus $x(t_e) = \frac{m}{c}\ln\left(\frac{v_0}{v_e}\right)$
Invullen levert: 1704 meter ofwel 1.7 km . . . een extra aanwijzing van het ongelofelijke verschil tussen klimmen en vlak rijden. De geinvesteerde energie om tot een steady state te komen waarbij je 388 Watt gebruikt brengt je minder dan 1 meter verder op een stevige helling maar 1700 meter op het vlakke.

Becijfering: Mgh argument met de 0.82 meter als uitkomst: 1 punt
Differentiaalvergelijking opstellen en analyse van de windkracht en =cv^2 1 punt
Uitwerken DV en oplossen: 1.5 punt.

Magnetic monopole

a) The Lorentz force law gives for the force on the electron

$$\vec{F} = q_e(\vec{v} \times \vec{B}) = q_e\left(\vec{v} \times \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{r}\right)$$

The acceleration is given by

$$\vec{a} = \frac{\vec{F}}{m} = \frac{\mu_0}{4\pi}\frac{q_e q_m}{m}\frac{(\vec{v} \times \vec{r})}{r^3}$$

b) We have to proof that $\frac{dT}{dt} = 0$.

Rewrite the kinetic energy as: $T = \frac{1}{2}m(\vec{v} \cdot \vec{v})$ and differentiate with respect to time,

$$\frac{dT}{dt} = \frac{1}{2}m\frac{d(\vec{v} \cdot \vec{v})}{dt} = m\vec{v} \cdot \frac{d\vec{v}}{dt} = m\vec{v} \cdot \vec{a} = \frac{\mu_0}{4\pi}\frac{q_e q_m}{r^3}\vec{v} \cdot (\vec{v} \times \vec{r}) = 0$$

c)

$$\frac{d|\vec{L}|^2}{dt} = \frac{d(\vec{L} \cdot \vec{L})}{dt} = 2\vec{L} \cdot \frac{d\vec{L}}{dt}$$

and

$$\frac{d\vec{L}}{dt} = m\frac{d(\vec{r} \times \vec{v})}{dt} = m(\vec{v} \times \vec{v}) + m(\vec{r} \times \vec{a}) = 0 + \vec{r} \times \vec{F} = \frac{\mu_0}{4\pi}\frac{q_e q_m}{r^3}\left(\vec{r} \times (\vec{v} \times \vec{r})\right)$$
$$= \frac{\mu_0}{4\pi}\frac{q_e q_m}{r^3}\left(\vec{v}(\vec{r} \cdot \vec{r}) - \vec{r}(\vec{r} \cdot \vec{v})\right)$$

Consequently,

$$\frac{d|\vec{L}|^2}{dt} = 2\vec{L} \cdot \frac{d\vec{L}}{dt} = 2\frac{\mu_0}{4\pi}\frac{q_e q_m}{r^3}\left(m(\vec{r} \times \vec{v})\right) \cdot \left(\vec{v}(\vec{r} \cdot \vec{r}) - \vec{r}(\vec{r} \cdot \vec{v})\right) = 0$$

Because $\vec{v} \perp (\vec{r} \times \vec{v})$ and $\vec{r} \perp (\vec{r} \times \vec{v})$

d)

$$\frac{d\vec{P}}{dt} = \frac{d\vec{L}}{dt} - \frac{\mu_0 q_e q_m}{4\pi}\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{\mu_0}{4\pi}\frac{q_e q_m}{r^3}\left(\vec{v}(\vec{r} \cdot \vec{r}) - \vec{r}(\vec{r} \cdot \vec{v})\right) - \frac{\mu_0 q_e q_m}{4\pi}\left(\frac{\vec{v}}{r} - \frac{\vec{r}}{r^2}\frac{dr}{dt}\right)$$
$$= \frac{\mu_0}{4\pi}\frac{q_e q_m}{r^3}\left(r^2\vec{v} - r^2\vec{v} - \vec{r}(\vec{r} \cdot \vec{v}) + r\vec{r}\frac{dr}{dt}\right) = \frac{\mu_0}{4\pi}\frac{q_e q_m}{r^3}\left(r\vec{r}\frac{d\sqrt{\vec{r} \cdot \vec{r}}}{dt} - \vec{r}(\vec{r} \cdot \vec{v})\right)$$
$$= \frac{\mu_0}{4\pi}\frac{q_e q_m}{r^3}\left(\frac{1}{2}\frac{1}{\sqrt{\vec{r} \cdot \vec{r}}}r\vec{r}\frac{d(\vec{r} \cdot \vec{r})}{dt} - \vec{r}(\vec{r} \cdot \vec{v})\right) = \frac{\mu_0}{4\pi}\frac{q_e q_m}{r^3}\left(\frac{1}{2}\frac{1}{r}r\vec{r}2(\vec{r} \cdot \vec{v}) - \vec{r}(\vec{r} \cdot \vec{v})\right)$$
$$= 0$$

e) Using the first hint we calculate $\vec{P} \cdot \hat{\varphi} = |\vec{P}|\hat{z} \cdot \hat{\varphi} = |\vec{P}|(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) \cdot \hat{\varphi} = 0$. On the other hand, we have

$$\vec{P}\cdot\hat{\varphi} = \left(\vec{L} - \frac{\mu_0 q_e q_m}{4\pi}\frac{\vec{r}}{r}\right)\cdot\hat{\varphi} = m(\vec{r}\times\vec{v})\cdot\hat{\varphi} - \frac{\mu_0 q_e q_m}{4\pi}(\hat{r}\cdot\hat{\varphi}) = m(\vec{r}\times\vec{v})\cdot\hat{\varphi}$$

Use

$\vec{v} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} + r\,\sin\theta\frac{d\varphi}{dt}\,\hat{\varphi}$ and $\vec{r} = r\hat{r}$, and it follows that

$$(\vec{r}\times\vec{v}) = r\left(\hat{r}\times\left(\frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} + r\,\sin\theta\frac{d\varphi}{dt}\,\hat{\varphi}\right)\right) = r^2\frac{d\theta}{dt}\hat{\varphi} - r^2\sin\theta\frac{d\varphi}{dt}\hat{\theta}$$

And thus

$$\vec{P}\cdot\hat{\varphi} = m(\vec{r}\times\vec{v})\cdot\hat{\varphi} = m\left(r^2\frac{d\theta}{dt}\hat{\varphi} - r^2\sin\theta\frac{d\varphi}{dt}\hat{\theta}\right)\cdot\hat{\varphi} = mr^2\frac{d\theta}{dt}$$

Combining the two expressions for $\vec{P}\cdot\hat{\varphi}$ leads to,

$$mr^2\frac{d\theta}{dt} = 0$$

and as this is valid for any r, we have

$$\frac{d\theta}{dt} = 0$$

so θ is a constant.

For the second part of the problem we use the second hint and consider

$$\vec{P}\cdot\hat{r} = |\vec{P}|(\hat{z}\cdot\hat{r}) = |\vec{P}|(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})\cdot\hat{r} = |\vec{P}|\cos\theta$$

And via the other route we have:

$$\vec{P}\cdot\hat{r} = \left(m(\vec{r}\times\vec{v}) - \frac{\mu_0 q_e q_m}{4\pi}\hat{r}\right)\cdot\hat{r} = -\frac{\mu_0 q_e q_m}{4\pi}$$

Finally,

$$\cos\theta = \cos\theta_P = -\frac{\mu_0 q_e q_m}{4\pi|\vec{P}|}$$

This result implies that the motion of the electron is confined to the surface of a cone that makes an angle θ_P with the z-axis of our new coordinate system.

f) We start with the constant kinetic energy T and use the fact that θ is constant so we can write,

$$\vec{v} = \frac{dr}{dt}\hat{r} + r\sin\theta_P\frac{d\varphi}{dt}\hat{\varphi}$$

and

$$T = \frac{1}{2}m|\vec{v}|^2 = \frac{1}{2}m\left(\frac{dr}{dt}\hat{r} + r\sin\theta_P\frac{d\varphi}{dt}\hat{\varphi}\right)\cdot\left(\frac{dr}{dt}\hat{r} + r\sin\theta_P\frac{d\varphi}{dt}\hat{\varphi}\right)$$
$$= \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\sin^2\theta_P\left(\frac{d\varphi}{dt}\right)^2\right)$$

We also can rewrite the angular momentum as

$$\vec{L} = m(\vec{r}\times\vec{v}) = m\left(r\hat{r}\times\left(\frac{dr}{dt}\hat{r} + r\sin\theta_P\frac{d\varphi}{dt}\hat{\varphi}\right)\right) = -mr^2\sin\theta_P\frac{d\varphi}{dt}\hat{\theta}$$

and

$$\left|\vec{L}\right|^2 = \vec{L}\cdot\vec{L} = m^2r^4\sin^2\theta_P\left(\frac{d\varphi}{dt}\right)^2$$

Substituting this in the equation for the energy to get rid of the φ-dependence we find,

$$T = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + \frac{\left|\vec{L}\right|^2}{m^2r^2}\right)$$

This is an equation of motion for the r-coordinate. Rewriting,

$$\left(\frac{dr}{dt}\right)^2 = \frac{2T}{m} - \frac{\left|\vec{L}\right|^2}{m^2r^2} \Rightarrow \frac{dr}{dt} = \pm\sqrt{\frac{2T}{m} - \frac{\left|\vec{L}\right|^2}{m^2r^2}}$$

The equation with the positive sign describes an electron moving out from the monopole, the negative sign describes the motion of an incoming electron. The expression under the square root should be larger or equal to zero leading to,

$$\frac{2T}{m} - \frac{\left|\vec{L}\right|^2}{m^2r^2} \geq 0 \Rightarrow r \geq r_m = \frac{\left|\vec{L}\right|}{\sqrt{2mT}}$$

This equation of motion can be solved by separation of variables and using a substitution $u = r^2$. For the outgoing electrons we find,

$$\int\frac{dr}{\sqrt{\frac{2T}{m} - \frac{\left|\vec{L}\right|^2}{m^2r^2}}} = \int dt$$

Using $u = r^2$ and thus $du = 2rdr$ and $dr = \frac{du}{2\sqrt{u}}$ leads to,

$$\int\frac{\frac{1}{2}du}{\sqrt{\frac{2Tu}{m} - \frac{\left|\vec{L}\right|^2}{m^2}}} = \int dt \Longrightarrow \int\frac{\frac{1}{2}\sqrt{\frac{m}{2T}}du}{\sqrt{u - \frac{2T\left|\vec{L}\right|^2}{m}}} = \int dt \Rightarrow \sqrt{u - \frac{\left|\vec{L}\right|^2}{2Tm}} = \sqrt{\frac{2T}{m}}t + const \Rightarrow$$

$$u - \frac{|\vec{L}|^2}{2Tm} = \left(\sqrt{\frac{2T}{m}}t + const\right)^2 \Rightarrow r^2 = r_m^2 + \left(\sqrt{\frac{2T}{m}}t + const\right)^2 \Rightarrow r^2 = r_m^2 + \frac{2T}{m}(t + t_0)^2$$

In a similar way we find for the incoming electrons:

$$u - \frac{|\vec{L}|^2}{2Tm} = \left(\sqrt{\frac{2T}{m}}t + const\right)^2 \Rightarrow r^2 = r_m^2 + \left(\sqrt{\frac{2T}{m}}t + const\right)^2 \Rightarrow r^2 = r_m^2 + \frac{2T}{m}(t_0 - t)^2$$

The incoming electrons will follow trajectories that spiral* over the cone defined by $\theta = \theta_P$ and move towards the monopole until they reach $r = r_m$, then they will bounce back and move spiralling outward over the cone. The radially directed magnetic field lines on the cone surface form a cage for the electron.

*For spiralling we still have to proof that $\frac{d\varphi}{dt} \neq 0$, this follows from

$$\left(\frac{d\varphi}{dt}\right)^2 = \frac{|\vec{L}|^2}{m^2 r^4 \sin^2\theta_P}$$

this is only zero if the initial angular momentum of the electron is zero, or in other words, if the electron is initially moving in the direction of the monopole at the origin.

Formation of a rainbow

Solution

1. We look at the angle, as they are defined by the Fig. 2.

$$\begin{aligned} D &= (AD, DF) = (AB, EF) \\ &= (AB, BC) + (BC, CE) + (CE, EF) \\ &= (i - r) + (\pi - 2r) + (i - r) \\ &= \pi + 2i4r \end{aligned}$$

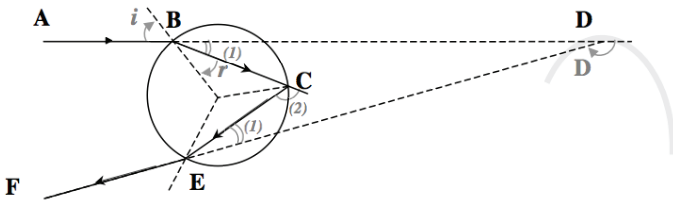


Figure 2: Calcul of the angle of Deviation D.

2. We are looking for the minimum of the function D for variable i. So we have to look at the zero of the derivatives:

$$\frac{dD}{di} = 2 - 4\frac{dr}{di} = 2\left(1 - 2\frac{dr}{di}\right) \tag{1}$$

The Descartes law tell us that $\sin i = n \sin r$ by derivation it gives us:

$$\cos i \,\mathrm{d}i = n \cos r \,\mathrm{d}r \tag{2}$$

therefore,

$$\frac{dr}{di} = \frac{\cos i}{n \cos r} \tag{3}$$

We finally obtain:

$$\frac{dD}{di} = 2\left(1 - \frac{2}{n}\frac{\cos i}{\cos r}\right) \tag{4}$$

The function D has extremum is $\frac{dD}{di} = 0$, so when:

$$\frac{\cos i}{\cos r} = \frac{n}{2} \tag{5}$$

We can notice that for $i = 0$, $\frac{dD}{di} < 0$ and for $i = \pi/2$, $\frac{dD}{di} > 0$. So we have a minimum when the conduction of Eq. 5 is fulfilled.
We have to express $\sin i$ in term of n. We use $\cos^2\theta + \sin^2\theta = 1$ and Eq. 5 becomes:

$$\frac{1 - \sin^2 i}{1 - \sin^2 r} = \frac{n^2}{4} \tag{6}$$

After some simplification, we obtain:

$$\sin i = \sqrt{\frac{1}{3}\left(4 - n^2\right)} \tag{7}$$

3. We have $\alpha = \pi - D = 4r - 2i$ and:

n	$\sin i$	$\sin r$	α
1.31	0.873	0.666	45.54°
1.33	0.862	0.648	42.51°

4 . This phenomena is called **dispersion**. Generally the index of dielectric depends on the wavelength. In our case, water, it is noticeable.

5. When the light is white, all colors are present, so all wavelength in the visible range are present. We have to see how the angle α depends on the wavelength λ. We have $\alpha = \pi - D = 4r - 2i$ so:

$$\mathrm{d}\alpha = 4\,\mathrm{d}r \tag{8}$$

for an incident angle i fixed. Once we derive the same way than earlier, we obtain that Eq. 8 can be rewritten as:

$$\mathrm{d}\alpha = -\frac{4}{n}\tan r\,\mathrm{d}n \tag{9}$$

Using Cauchy's formula, we have $n = n_0 + \frac{C}{\lambda_0^2} \rightarrow \mathrm{d}n = -\frac{2C\,\mathrm{d}\lambda_0}{\lambda_0^3}$. Therefore, Eq. 9 becomes:

$$\mathrm{d}\alpha = +\frac{8C}{n\lambda_0^3}\tan r\,\mathrm{d}\lambda_0 \tag{10}$$

This last equation shows that when the wavelength λ_0 increases the angle α also increases. The different arcs of concentric circles have color violet inside(λ & α and "smaller") inside red outside (λ & α "larger").

6. The phenomena occurs such that the observer sees it with the same angle, therefore the result is a cone of view. We only see the top part of the cone, the other being hidden by the ground (unless you are on top of a mountain and look in the valley).

7. You should be standing first with the sun on your back and facing the curtain of rain (see Fig. 3). You should look up at an angle of $\sim 110°$ above he horizon to see the top of the rainbow

   The height h where you see the top of the rainbow depends on the distance you are from the rain curtain d: $h = \frac{d}{\tan(D/2)}$. If you are 1 km from the rain, you will see it 363 m above the horizon.

8. No you will never be able to reach it. When you come closer to the curtain of rain, the rainbow is still visible but moves downwards and get smaller. Once you reach the curtain you will still see the *origin* of the rainbow on your side.

9. It is possible to have a second rainbow due to possible second reflection in the droplet (see the trajectory in Fig. 4). It should be placed below the first one, inverted in color because of the second reflection, follows the first one (cone with a different angle). We say sometimes cause the intensity is weaker than the first rainbow, and it is more difficult to see.

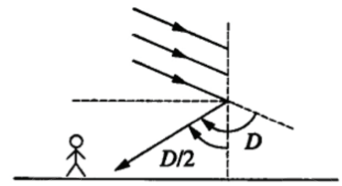


Figure 3: Where should you be and where should you look to see the rainbow.

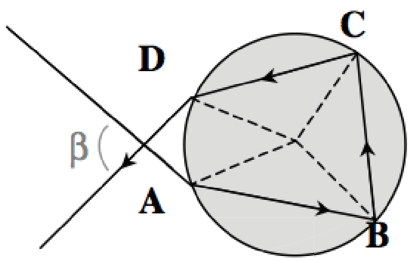


Figure 4: Schematic figure of the trajectory of a sun ray in a spherical water droplet for the second reflection

10. We obtain the new deviation angle via: $D_2 = (i-r)+2(\pi-2r)+(i-r) = 2\pi+2i6r$. We write $\beta = D_2-\pi = 2(i-3r)$. We use the same reasoning that the first time. Determine the $\sin i$ as a function of n then look for the minimum of deviation. We have:

$$\frac{dD}{di} = 2\left(1-3\frac{dr}{di}\right) = 2\left(1-3\frac{\cos i}{n\cos r}\right) \quad (11)$$

Once we use Descartes law, we finally obtain:

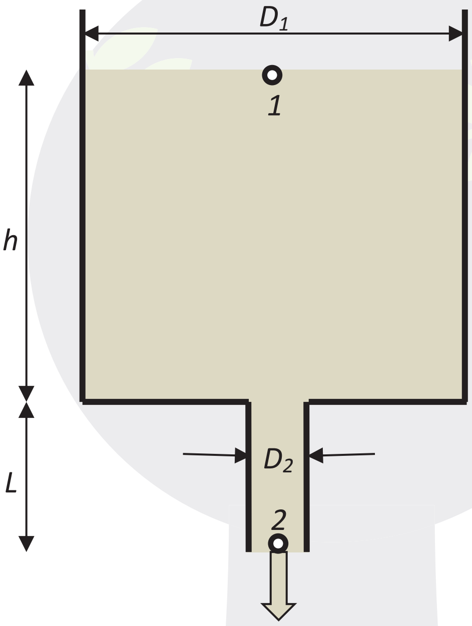
$$\sin i = \sqrt{\frac{1}{8}\left(9-n^2\right)} \quad (12)$$

For water, $\beta = 50.1°$.
We continue in looking the way β changes with the wavelength λ_0:

$$\mathrm{d}\beta = -6dr = +6\frac{6}{n}\tan r\,\mathrm{d}n = -\frac{12C}{n\lambda_0^3}\tan r\,\mathrm{d}\lambda_0 \quad (13)$$

The function $\beta(\lambda_0)$ is decaying for $[0;\pi/2]$ so if λ_0 increases, β will decrease. The Rainbow is then inverted.

Draining a barrel



a) Mechanische Energiebalans (Bernoulli, uitgebreid met wrijving)

$$0=\frac{P_1-P_2}{\rho}+\frac{v_1^2-v_2^2}{2}+g\left(z_1-z_2\right)-e_{fr}$$

$$\left.\begin{array}{l}\rightarrow v_2^2=2g\left(z_1-z_2\right)-2e_{fr}\\ e_{fr}=cLv_2^2\\ z_1-z_2=h+L\end{array}\right\}v_2=\frac{\sqrt{2g}\sqrt{h+L}}{\sqrt{1+2cL}}$$

massabalans over het vat: $\frac{\pi}{4}D_1^2\frac{dh}{dt}=-\frac{\pi}{4}D_2^2 v_2$

$$\frac{dh}{dt}=-\left(\frac{D_2^2}{D_1^2}\sqrt{2g}\right)\frac{\sqrt{h+L}}{\sqrt{1+2cL}}$$

$$\left.\begin{array}{l}\frac{dh}{dt}=-\left(\frac{D_2^2}{D_1^2}\sqrt{2g}\right)\frac{\sqrt{h+L}}{\sqrt{1+2cL}}\\ t=0:h=h_0\end{array}\right\}\left(h_0+L\right)^{1/2}-\left(h+L\right)^{1/2}=\frac{D_2^2}{D_1^2}\sqrt{\frac{g}{2}}\frac{1}{\left(1+2cL\right)^{1/2}}t$$

vat leeg op $t=\tau\rightarrow$ $\boxed{\tau=\sqrt{\frac{2}{g}}\frac{D_1^2}{D_2^2}\left[\left(h_0+L\right)^{1/2}-L^{1/2}\right]\left(1+2cL\right)^{1/2}}$

b) $\tau_{L=0}=\sqrt{\frac{2}{g}}\frac{D_1^2}{D_2^2}h_0^{1/2}$

$$\tau_{L\to\infty}=\lim_{L\to\infty}\sqrt{\frac{2}{g}}\frac{D_1^2}{D_2^2}L^{1/2}\left[\left(1+\frac{h_0}{L}\right)^{1/2}-1\right]\left(1+2cL\right)^{1/2}\approx\sqrt{\frac{2}{g}}\frac{D_1^2}{D_2^2}L^{1/2}\frac{h_0}{2L}\sqrt{2c}L^{1/2}=\sqrt{\frac{2}{g}}\frac{D_1^2}{D_2^2}\frac{h_0c^{1/2}}{\sqrt{2}}$$

c) $\frac{d\tau}{dL}=0\rightarrow\frac{2}{K}(1+2cL)^{1/2}\frac{1}{2}\left[(h_0+L)^{-1/2}-L^{-1/2}\right]+\frac{2}{K}\left[(h_0+L)^{1/2}-L^{1/2}\right]\frac{1}{2}(1+2cL)^{-1/2}2c=0$

$$\rightarrow(1+2cL)\left[(h_0+L)^{-1/2}-L^{-1/2}\right]+\left[(h_0+L)^{1/2}-L^{1/2}\right]2c=0$$

$$\rightarrow(1+2cL)\left[(h_0+L)^{-1/2}-L^{-1/2}\right]\left[(h_0+L)^{1/2}+L^{1/2}\right]+2ch_0=0$$

$$\rightarrow(1+2cL)\left[\left(\frac{L}{h_0+L}\right)^{1/2}-\left(\frac{h_0+L}{L}\right)^{1/2}\right]+2ch_0=0$$

$$\rightarrow\left[\left(\frac{h_0+L}{L}\right)^{1/2}-\left(\frac{L}{h_0+L}\right)^{1/2}\right]=2\frac{ch_0}{1+2cL}\rightarrow\frac{h_0}{L}=2\frac{ch_0}{1+2cL}\left(\frac{h_0+L}{L}\right)^{1/2}$$

$$\rightarrow L=\frac{1+2cL}{2c}\left(\frac{h_0+L}{L}\right)^{-1/2}\rightarrow L^2=\frac{1+4cL+4c^2L^2}{4c^2}\left(\frac{L}{h_0+L}\right)\rightarrow 4c^2L(h_0+L)=1+4cL+4c^2L^2$$

$\rightarrow$ $\boxed{L_{opt}=\frac{1}{4c(ch_0-1)}}$. Er bestaat dus een optimale slanglengte indien $h_0>c^{-1}$ (of $c>h_0^{-1}$)

d) $\tau=\sqrt{\frac{2}{g}\frac{D_1^2}{D_2^2}}\left[(h_0+L)^{1/2}-L^{1/2}\right](1+2cL)^{1/2}$

$$\tau_{L=L_{opt}}=\sqrt{\frac{2}{g}\frac{D_1^2}{D_2^2}}\left[\left(h_0+\frac{1}{4c(ch_0-1)}\right)^{1/2}-\left(\frac{1}{4c(ch_0-1)}\right)^{1/2}\right]\left(\frac{2(ch_0-1)+1}{2(ch_0-1)}\right)^{1/2}$$

$$=\sqrt{\frac{2}{g}\frac{D_1^2}{D_2^2}}\frac{1}{(ch_0-1)}\left[\left(\frac{4c^2h_0^2-4ch_0+1}{4c}\right)^{1/2}-\left(\frac{1}{4c}\right)^{1/2}\right]\left(\frac{2(ch_0-1)+1}{2}\right)^{1/2}$$

$$=\sqrt{\frac{2}{g}\frac{D_1^2}{D_2^2}}\frac{1}{(ch_0-1)2\sqrt{c}}\left[2ch_0-2\right]\left(\frac{2(ch_0-1)+1}{2}\right)^{1/2}$$

$$\rightarrow\boxed{\tau_{L=L_{opt}}=\sqrt{\frac{2}{g}\frac{D_1^2}{D_2^2}}\left(h_0-\frac{1}{2c}\right)^{1/2}}$$

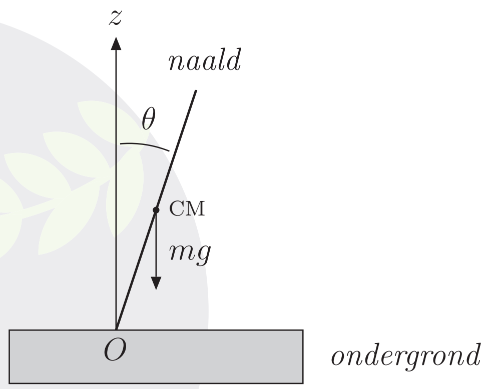
$$\tau_{L=L_{opt}}=\sqrt{\frac{2}{g}\frac{D_1^2}{D_2^2}}\left(h_0-\frac{1}{2c}\right)^{1/2}<\sqrt{\frac{2}{g}\frac{D_1^2}{D_2^2}}h_0^{1/2}=\tau_{L=0}$$

$$\tau_{L=L_{opt}}=\sqrt{\frac{2}{g}\frac{D_1^2}{D_2^2}}\left(h_0-\frac{1}{2c}\right)^{1/2}=\sqrt{\frac{2}{g}\frac{D_1^2}{D_2^2}}\frac{h_0c^{1/2}}{\sqrt{2}}\left(1-\left(\frac{h_0c-1}{h_0c}\right)^2\right)^{1/2}<\sqrt{\frac{2}{g}\frac{D_1^2}{D_2^2}}\frac{h_0c^{1/2}}{\sqrt{2}}=\tau_{L\rightarrow\infty}$$

dus het extremum bij $L=L_{opt}$ is een minimum

Precair evenwicht

Stel we willen een lange naald met de punt naar beneden op een vlakke, harde ondergrond laten balanceren. De ondergrond mag natuurlijk niet te glad zijn om wegglijden te voorkomen en het experiment dient in een trillingsvrije ruimte bij een zo laag mogelijke temperatuur te worden uitgevoerd. We hebben aan alles gedacht. Echt alles?



Nee, één tak van de fysica blijft moedig weerstand bieden ... de kwantummechanica!

(a) We beginnen met de bepaling van het traagheidsmoment I van de naald behorende bij de valbeweging. De valbeweging is een rotatiebeweging rond de punt O van de naald, beschreven in termen van de hoek θ tussen de naald en de verticale stand (zie plaatje). Vat de naald bij benadering op als een 1-dimensionaal staafje met homogene massaverdeling, lengte ℓ en totale massa m. De massa per lengte-eenheid van de naald wordt dus gegeven door m/ℓ en het gezochte traagheidsmoment door

$$I = \int_0^\ell \mathrm{d}r \, \frac{m}{\ell} \, r^2 = \frac{1}{3} m\ell^2 \,. \tag{1}$$

(b) Klassiek: beschouw de valbeweging van de naald eerst als zijnde klassiek. De Lagrangiaan behorende bij deze beweging bestaat uit een rotationele kinetische energieterm

$$T_{\rm rot} = \frac{I}{2}\dot{\theta}^2 = \frac{1}{6} m\ell^2\dot{\theta}^2$$

en een gravitationele potentiaalterm ten gevolge van de verticale positie van het zwaartepunt (CM)

$$V = mg\, z_{\rm CM} = \frac{1}{2} mg\ell\cos\theta \,,$$

waarbij g de gravitationele valversnelling is en $\dot{\theta} \equiv \mathrm{d}\theta/\mathrm{d}t$ de hoeksnelheid behorende bij de valbeweging. Uit de Lagrangiaan

$$L(\theta,\dot{\theta}) = T_{\rm rot} - V = \frac{1}{6} m\ell^2\dot{\theta}^2 - \frac{1}{2} mg\ell\cos\theta$$

volgt dan de bewegingsvergelijking behorende bij de valbeweging:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial\dot{\theta}}\Big) - \frac{\partial L}{\partial\theta} = \frac{1}{3} m\ell^2\ddot{\theta} - \frac{1}{2} mg\ell\sin\theta = 0$$

$$\Rightarrow \quad \ddot{\theta}(t) = \frac{3g}{2\ell}\sin(\theta(t)) \equiv \omega^2\sin(\theta(t)) \qquad \text{met} \qquad \omega = \sqrt{\frac{3g}{2\ell}} \,.$$

Voor kleine hoeken wordt dit bij benadering

$$\ddot{\theta}(t) \approx \omega^2 \theta(t) \quad \Rightarrow \quad \theta(t) \approx A\,\mathrm{e}^{\omega \mathrm{t}} + \mathrm{B}\,\mathrm{e}^{-\omega \mathrm{t}} \,, \qquad (2)$$

waarbij de constanten A en B volgen uit de begincondities $\theta(t=0) = \theta_0 \geq 0$ en $\dot{\theta}(t=0) = \omega_0 \geq 0$:

$$A = \tfrac{1}{2}\,(\theta_0 + \omega_0/\omega) \quad \text{en} \quad B = \tfrac{1}{2}\,(\theta_0 - \omega_0/\omega) \,. \qquad (3)$$

(c) Kwantum: idealiter zou je de naald op tijdstip $t = 0$ zodanig met de punt naar beneden op de vlakke ondergrond willen zetten dat zowel de initiële hoek θ_0 met de verticale stand als de bijbehorende initiële hoeksnelheid ω_0 nul zijn. Echter, de onzekerheidsrelatie van Heisenberg laat dit niet toe. De onzekerheidsrelatie van Heisenberg zegt nu dat $\Delta q\,\Delta p_q \geq \hbar/2$, waarbij Δq de kwantummechanische onzekerheid is in de gegeneraliseerde coördinaat q en Δp_q de onzekerheid in de bijbehorende impuls $p_q = \partial L/\partial \dot{q}$. Hoe deze kwantummechanische restrictie precies moet worden ingezet is niet zo eenvoudig te formuleren, maar neem aan dat we het simpelweg kunnen loslaten op de beginsituatie op $t = 0$ door te eisen dat $\theta(0)\,p_\theta(0) = \theta_0 I \omega_0 \geq \hbar/2$.

We gaan nu een schatting maken voor de maximale tijd $\tau_{\max}$ die de naald erover doet om de balansverstorende hellingshoek $\theta_M = 0.1$ radialen te bereiken, uitgaande van een beginsituatie die perfect klassiek evenwicht (d.w.z. $\theta_0 = \omega_0 = 0$) zo dicht mogelijk benadert. De afvallende e-macht in (2) is en blijft zeer klein voor alle $t \geq 0$. De groeiende e-macht $\propto A$ kan daarentegen uitgroeien tot balansverstorende proporties als

$$A\,\mathrm{e}^{\omega \mathrm{t}} \geq \theta_{\mathrm{M}} \quad \Rightarrow \quad \mathrm{t} \geq \frac{1}{\omega}\,\ln(\theta_{\mathrm{M}}/\mathrm{A}) \,.$$

De hiervoor benodigde tijd is maximaal als $A = \frac{1}{2}\,(\theta_0 + \omega_0/\omega) \geq 0$ minimaal is, mits dit minimum consistent is met de onzekerheidsrelatie van Heisenberg. Op grond van de onzekerheidsrelatie weten we dat

$$A \geq \frac{1}{2}\left(\theta_0 + \frac{\hbar}{2I\omega}\,\frac{1}{\theta_0}\right) \quad \Rightarrow \quad A_{\min} = \sqrt{\frac{\hbar}{2I\omega}} = \sqrt{\frac{\hbar\omega}{m g \ell}} \,.$$

De maximale houdbaarheidsduur van het evenwicht is dus

$$\tau_{\max} \approx \frac{1}{\omega}\,\ln\!\left(\theta_M \sqrt{\frac{m g \ell}{\hbar\omega}}\right) . \qquad (4)$$

Met behulp van de numerieke input $m = 0.01\,\mathrm{kg}$, $\ell = 0.1\,\mathrm{m}$, $\mathrm{g} = 9.81\,\mathrm{m\,s^{-2}}$ en $\hbar = 1.055 \times 10^{-34}\,\mathrm{kg\,m^2\,s^{-1}}$ vinden we dan dat

$$\boxed{\tau_{\max} \approx 2.74\,\mathrm{s}} \,,$$

hetgeen verrassend alledaags is!

(d) Thermische invloed: de laagste temperaturen die we ooit in het laboratorium hebben kunnen realiseren liggen in het $0.1\,\mathrm{nK}$ bereik. Neem aan dat we de naald tot zo'n temperatuur hebben kunnen afkoelen. Vervolgens passen we het klassieke principe van equipartitie van energie toe op de rotationele vrijheidsgraad van de naald, aangezien $p_\theta = I\dot{\theta}$ kwadratisch in de kinetische energie voorkomt. In thermisch evenwicht met een warmtebad bij temperatuur T heeft de naald dan een gemiddelde rotationele energie

$$\frac{I}{2}\langle\dot{\theta}^2\rangle = \frac{1}{2}kT \quad \Rightarrow \quad \langle\dot{\theta}^2\rangle = \frac{kT}{I} \,.$$

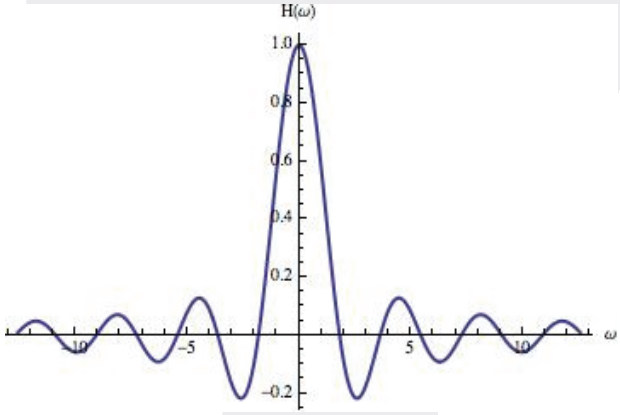
Dit leidt tot een $\mathcal{O}(\sqrt{\frac{kT}{I}})$ thermische bijdrage aan de initiële hoeksnelheid ω_0. De kwantummechanische bijdrage aan ω_0 in onderdeel (c) was $\mathcal{O}(\sqrt{\frac{\hbar\omega}{I}})$. Om de kwantummechanische effecten uit onderdeel (c) te kunnen waarnemen zouden we de naald dus moeten afkoelen tot $T_{\mathrm{QM}} = \mathcal{O}(\hbar\omega/k)$. Gebruik makende van de numerieke input uit onderdeel (c), alsmede $k = 1.381 \times 10^{-23}\,\mathrm{kg\,m^2\,s^{-2}\,K^{-1}}$, vinden we dat $T_{\mathrm{QM}} = \mathcal{O}(0.1\,\mathrm{nK})$ weldegelijk precies in het haalbare bereik ligt!

Filtering by many identical systems

a) (1 point) The Fourier transform is given by:

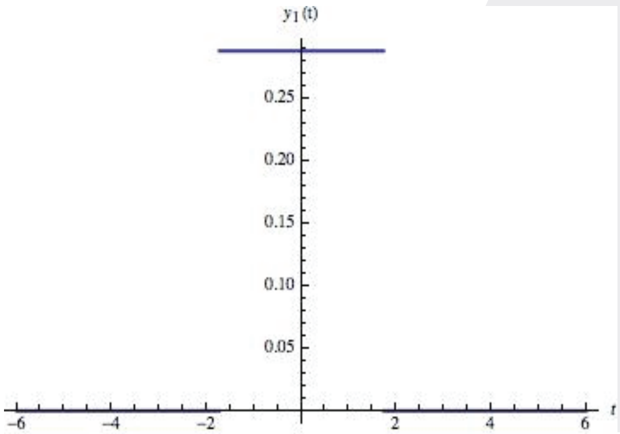
$$H(\omega)=\int_{-\infty}^{+\infty} h(t)e^{-i\omega t}\,dt=\int_{-\sqrt{3}}^{+\sqrt{3}} \frac{1}{2\sqrt{3}}e^{-i\omega t}\,dt=\left(\frac{1}{2\sqrt{3}}\right)\frac{e^{-i\omega\sqrt{3}}-e^{+i\omega\sqrt{3}}}{-i\omega}=\frac{\sin(\sqrt{3}\omega)}{\sqrt{3}\omega}$$

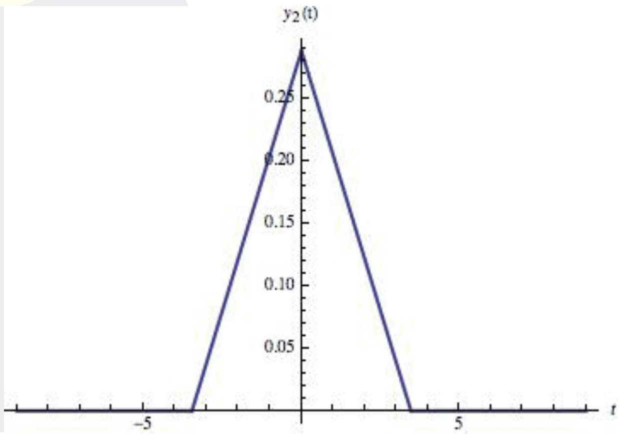
Note that H(ω = 0) = 1.



b) (2 points) The signals $y_1(t)$ and $y_2(t)$ are given by:

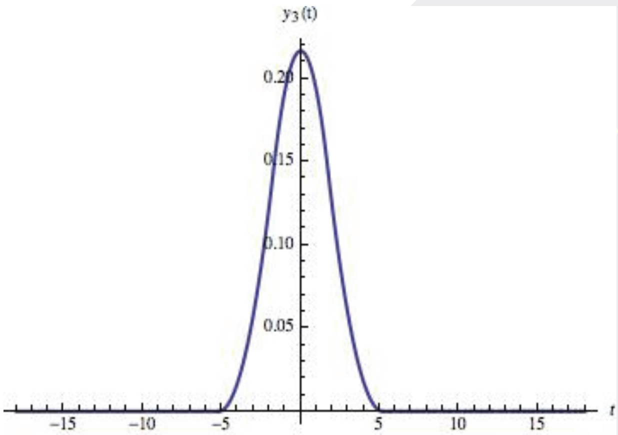
$$y_1(t)=h(t) \qquad y_2(t)=\begin{cases}\frac{1}{12}(2\sqrt{3}-|t|) & |t|\le 2\sqrt{3}\\ 0 & |t|>2\sqrt{3}\end{cases}$$

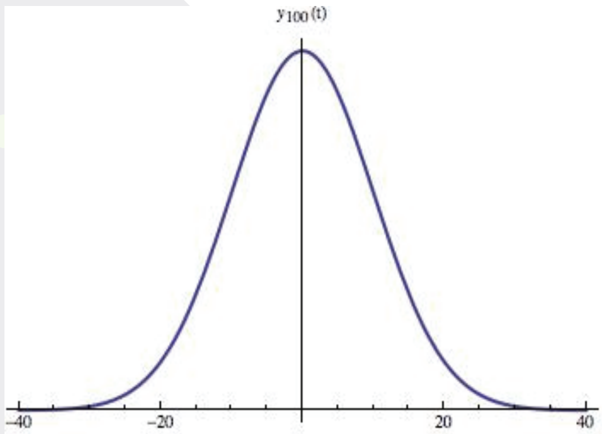




c) (2 points) The signals $y_3(t)$ and $y_{100}(t)$ are given by:

$$y_{100}(t) \approx \frac{1}{10\sqrt{2\pi}} e^{-\left(t^2/2\bullet 100\right)}$$





d) (3 points) The repeated convolution with the rectangular impulse response leads tc result that increasingly resembles a Gaussian signal. This is an example of the **Centr Limit Theorem**. Remember that the sum of two independent, identically distribute random variables has a probability density function that can be described by t convolution of the density function of one variable with itself. That is $y_2(t)$. Repeat convolutions lead to the convergence that is the Central Limit Theorem. This problem the Drunkard's Walk.

e) (6 points) It is easy to show that the center, y_c, of every one of the signal is zero, that $\mathbf{y_{Nc} = 0}$. Here are two proofs. The second may seem indirect—and is—but will come handy.

Proof #1:
Starting from

$$y_c = \frac{\int_{-\infty}^{+\infty} t y(t)\,dt}{\int_{-\infty}^{+\infty} y(t)\,dt}$$

we see that the denominator is just some positive number in this case 1. The signal y(t) is even anc the signal "t" is odd. The product is, therefore, odd. The area (integral) of an odd function over a symmetric interval is therefore zero.

For $y_2(t)$ and $y_N(t)$ the results of the convolutions remain even so the center remains at t = 0, that is $\mathbf{y_{Nc} = 0}$ **for all N ≥ 1**.

Proof #2:
Starting from

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-i\omega t}\,dt$$

we see that:

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{+\infty} -itx(t)e^{-i\omega t}\,dt \quad \Rightarrow \quad \int_{-\infty}^{+\infty} tx(t)\,dt = i\left.\frac{dX(\omega)}{d\omega}\right|_{\omega=0}$$

$$\Rightarrow x_c = \frac{i\left.\frac{dX(\omega)}{d\omega}\right|_{\omega=0}}{X(\omega=0)}$$

Using $y_1(t) = h(t)$ means $Y_1(\omega) = H(\omega)$:

$$y_{1c} = \frac{i\left.\frac{dH(\omega)}{d\omega}\right|_{\omega=0}}{H(\omega=0)} = \frac{i\left(\frac{3\omega\cos(\sqrt{3}\omega)-\sqrt{3}\sin(\sqrt{3}\omega)}{3\omega^2}\right)}{1} = i\left(\frac{3\omega\cos(\sqrt{3}\omega)-\sqrt{3}\sin(\sqrt{3}\omega)}{3\omega^2}\right)$$

Using L'Hôpital's rule to evaluate this gives $\mathbf{y_{1c} = 0}$. Using this approach for $y_N(t)$ means $Y_N(\omega) = H^N(\omega)$. It follows from the chain rule that:

$$\frac{dY_N(\omega)}{d\omega} = \frac{dH^N(\omega)}{d\omega} = \left(\frac{dH^N(\omega)}{dH}\right)\left(\frac{dH(\omega)}{d\omega}\right) = N\,H^{N-1}(\omega)\left(\frac{dH(\omega)}{d\omega}\right)$$

Evaluating this for $\omega = 0$ means $H^{N-1}(\omega = 0) = 1$ and $dH/d\omega = 0$. Thus $\mathbf{y_{Nc} = 0}$ **for all** $\mathbf{N \geq 1}$.
Now let us look at the rms width. The numerical value of the width can also be determined in two ways.

Proof #1:
Starting from the definition and the results that $y_{Nc} = 0$ and the *area* under $y_1(t) = h(t) = 1$, we have:

$$y_{rms} = \sqrt{\frac{\int_{-\infty}^{+\infty}(t-y_c)^2\,y(t)\,dt}{\int_{-\infty}^{+\infty}y(t)\,dt}} = \sqrt{\int_{-\infty}^{+\infty}t^2 y(t)\,dt}$$

Direct numerical evaluation of the width of $y_1(t)$ follows.

$$y_{1,rms} = \sqrt{\int_{-\infty}^{+\infty}t^2 y(t)\,dt} = \sqrt{\int_{-\sqrt{3}}^{+\sqrt{3}}t^2\left(\frac{1}{2\sqrt{3}}\right)dt} = 1$$

For the width of $y_N(t)$ for $N > 1$, we will appeal to the Fourier method..

Proof #2:

Starting from the definition and the results that $y_{Nc} = 0$ and the *area* under $y_1(t) = h(t) = 1$, we have:

$$y_{rms} = \sqrt{\frac{\int_{-\infty}^{+\infty} (t - y_c)^2 y(t)\,dt}{\int_{-\infty}^{+\infty} y(t)\,dt}} = \sqrt{\int_{-\infty}^{+\infty} t^2 y(t)\,dt} = \sqrt{\int_{-\infty}^{+\infty} t^2 h(t)\,dt}$$

Going further with the Fourier transforms gives:

$$-\frac{d^2 H(\omega)}{d\omega^2} = \int_{-\infty}^{+\infty} t^2 h(t) e^{-i\omega t}\,dt \Rightarrow \qquad \int_{-\infty}^{+\infty} t^2 h(t)\,dt = -\left.\frac{d^2 H(\omega)}{d\omega^2}\right|_{\omega=0}$$

$$\Rightarrow y_{rms} = \sqrt{\frac{-\left.\frac{d^2 H(\omega)}{d\omega^2}\right|_{\omega=0}}{H(\omega = 0)}} = \sqrt{-\left.\frac{d^2 H(\omega)}{d\omega^2}\right|_{\omega=0}}$$

The Fourier spectrum can give us the rms width of the signal. But what is $d^2H/d\omega^2$? A bit nasty but you only have to do it once (correctly):

$$\frac{d^2 H(\omega)}{d\omega^2} = \frac{-6\omega\cos(\sqrt{3}\omega) + 2\sqrt{3}\sin(\sqrt{3}\omega) - 3\sqrt{3}\omega^2\sin(\sqrt{3}\omega)}{3\omega^3}$$

Using L'Hôpital's rule, again, gives $\mathbf{y_{1,rms} = 1}$. And this agrees with the value from Proof #1. And now for N > 1. We can write the derivatives based upon the chain rule and the $Y_N(\omega) = H^N(\omega)$ as:

$$\frac{d^2 Y_N(\omega)}{d\omega^2} = \frac{d}{d\omega}\left(\frac{dY_N(\omega)}{d\omega}\right) = \frac{d}{d\omega}\left(\frac{dH^N(\omega)}{d\omega}\right) = \frac{d}{d\omega}\left(N H^{N-1}\frac{dH}{d\omega}\right)$$

$$= \left(N(N-1)H^{N-2}\left(\frac{dH}{d\omega}\right)^2\right) + \left(N H^{N-1}\frac{d^2 H}{d\omega^2}\right)$$

This may seem formidable but we have already shown that $dH/d\omega = 0$ when $\omega \to 0$ meaning that the first term vanishes leaving only the second term. We now evaluate for $\omega \to 0$:

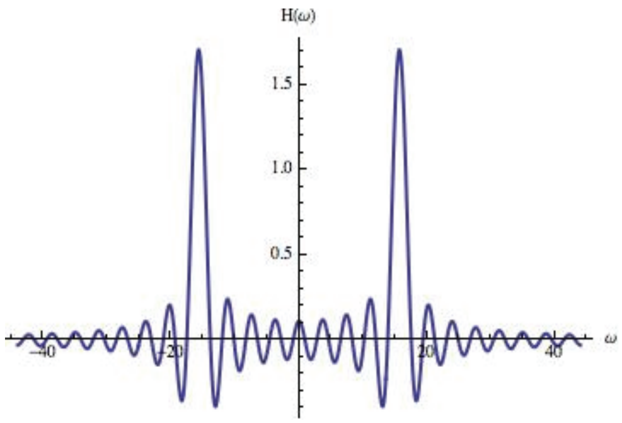
$$\left.\frac{d^2 Y_N(\omega)}{d\omega^2}\right|_{\omega=0} = N H^{N-1}\left.\frac{d^2 H}{d\omega^2}\right|_{\omega=0} = N(1)^{N-1}(-1) = -N$$

This means that $y_{N,rms} = \sqrt{N}$ for all N. Note, again, how this coincides with the characteristics of the Drunkard's Walk in diffusion theory.

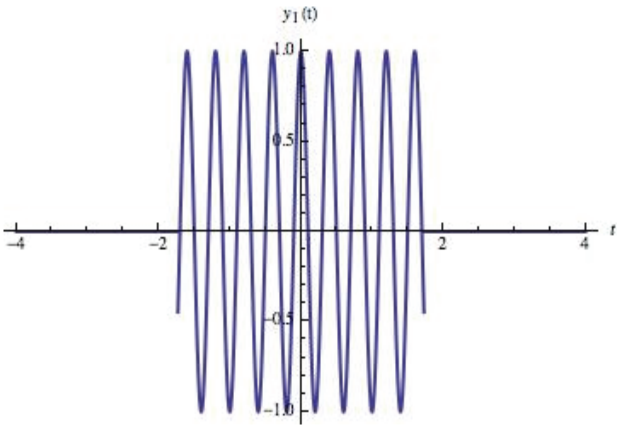
f) (2 points) The new impulse response is $h(t) = \cos(5\pi t)\left(u\left(t+\sqrt{3}\right) - u\left(t-\sqrt{3}\right)\right)$.

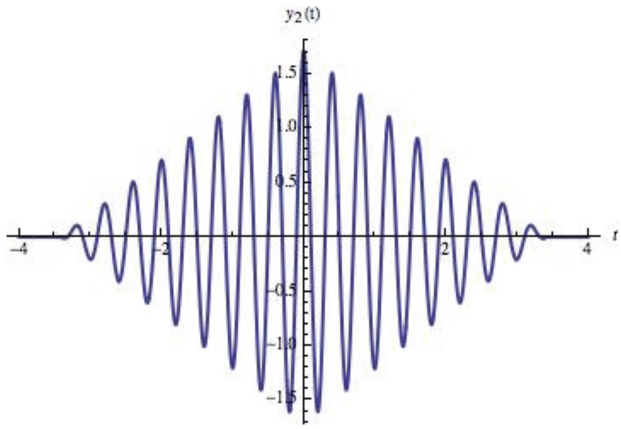
This multiplication in the time domain is a convolution in the frequency domain yielding:

$$H(\omega) = \frac{\sin\left(\sqrt{3}(5\pi-\omega)\right)}{(5\pi-\omega)} + \frac{\sin\left(\sqrt{3}(5\pi+\omega)\right)}{(5\pi+\omega)}$$



g) (2 points) The signals $y_1(t)$ and $y_2(t)$ are given by:

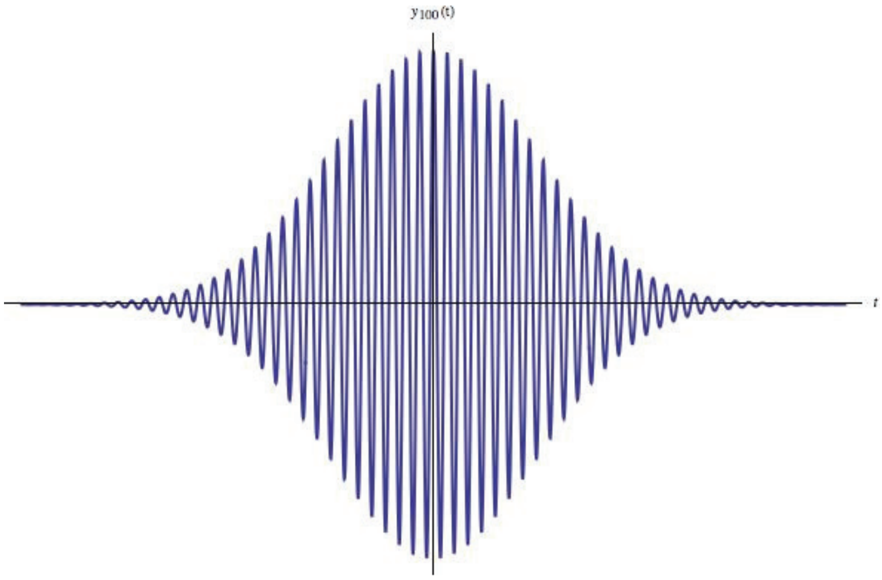




The carrier frequency, $\omega_o = 5\pi$, remains the same but the envelope has the shapes that we found in the first section of this problem. It should be clear that this is due to the $H^N(\omega)$ behavior after N filters.

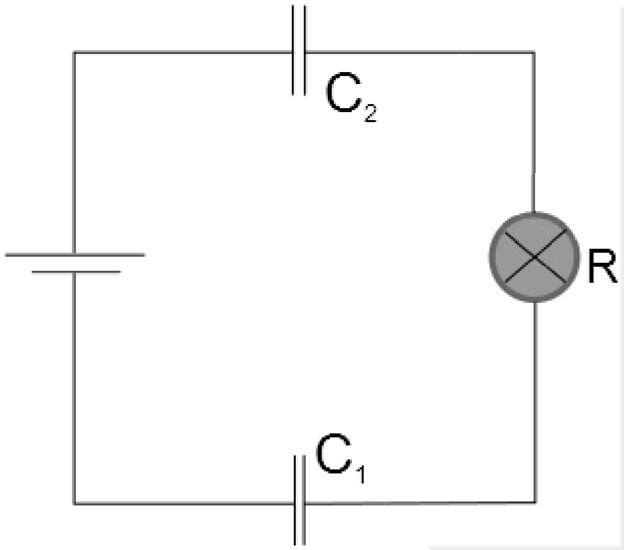
h) (2 points) The signal $y_{100}(t)$—again as a result of the Central Limit Theorem—is given by:

$$y_{100}(t) \approx \frac{\cos(5\pi t)}{10\sqrt{2\pi}} e^{-\left(t^2/2\bullet 100\right)}$$



The carrier frequency, $\omega_o = 5\pi$, remains the same but the envelope now has a Gaussian shape with $\sigma = \sqrt{N} = 10$.

Bouncing Battery



a) We assume the two battery contacts are parallel-plate capacitors with time-varying separations $x_1(t)$and $x_2(t) = d - x_1(t)$. Their capacitance is given by

$$C_{1,2}(t) = \frac{\epsilon A}{x_{1,2}(t)}$$

With $Q = CV$ and Kirchhoff's law we have

$$V_{batt} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} + IR$$

Since charge is conserved and the whole system is electrically neutral we have that the charges of the two capacitors are equal: $Q_1 = Q_2$, so

$$V_{batt} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) + IR = Q\left(\frac{x_1(t)}{\epsilon A_1} + \frac{d - x_1(t)}{\epsilon A_2}\right) + IR$$

With $I \equiv \frac{dQ}{dt}$ this gives us a differential equation for Q:

$$\frac{dQ}{dt}R = IR = V_{batt} - Q\left(\frac{x_1(t)}{\epsilon A_1} + \frac{d - x_1(t)}{\epsilon A_2}\right)$$

Now there are two cases:

1) For $A_1 = A_2$ (areas of battery contacts) the total capacitance becomes time-independent and the circuit reduces to a battery, resistance and capacitor connected in series – which will simply charge the capacitor to

$$Q = \frac{\epsilon A V_{batt}}{d}$$

and sustain no current.

2) For $A_1 \neq A_2$we can assume the current is very small: $IR \ll V_{batt}$ which reduces the differential equation to

$$Q = \frac{V_{batt}\epsilon A_2}{d + \frac{A_1 - A_2}{A_1} x_1(t)}$$

The current is now simply the time derivative of the above expression:

$$I = \frac{dQ}{dt} = -\frac{V_{batt}\epsilon A_2}{\left(d + \frac{A_1 - A_2}{A_1} x_1(t)\right)^2} \frac{A_1 - A_2}{A_1} \frac{d}{dt} x_1(t)$$

b) With$\epsilon = 8.85 * 10^{-12}$,$A_2 = 0.5\, A_1 = 0.5 \frac{\pi}{4} (0.01m)^2$ and $d = 1mm$ the expression for Q reduces to

$$Q \approx 10^{-12} \frac{1}{1 + 0.5 \frac{x_1(t)}{d}}$$

now $x_1(t)/d$ varies between 0 and 1 so the last factor varies between 1 and 0.66. Therefore

$$I = \frac{dQ}{dt} \approx 10^{-12} * 0.33 * 20 \approx 6 * 10^{-12}$$

and

$$P = I^2 R \approx 10^{-22}\, W$$

Temperature in a finite system

(1) (a) The total momentum is conserved because of translation symmetry.

(b) There appear to be $3N$ momentum coordinates. However, due to total momentum conservation, there are only $3N-3$ actual momentum degrees of freedom. Hence

$$T = \frac{2K}{3N-3}.$$

(c) We have

$$\frac{1}{T} = k_B \frac{1}{\Omega}\frac{\partial \Omega}{\partial E}$$

where we keep the volume and particle number constant in taking the derivative.
We start from

$$\Omega = \frac{1}{h^{3N-3}N!}\int \delta\left[E - H(p,q)\right] d^{3N-3}p\, d^{3N}q.$$

If we first integrate over the momenta, we note that they should all lie on a hypersphere centered at the origin, defined by $K = E - V(q)$, where E is the total energy, K is the kinetic and V the potential energy. Therefore the integration element over the $3N-3$ dimensional coordinate p can be written as

$$d^{3N-3}p = \omega(3N-3)P^{3N-4}\,dP$$

where $P = \sqrt{\sum_i p_i^2}$, with $\sum_i$ a sum over the particles. The kinetic energy is then

$$K = \frac{P^2}{2m}.$$

From this, we immediately have

$$P^{3N-4}dP = (2m)^{(3N-3)/2}K^{(3N-5)/2}dK/2$$

Replacing K by $E-V$ as imposed by the delta-function directly leads to

$$\Omega = \frac{\omega(3N-3)(2m)^{(3N-3)/2}}{2h^{3N-3}N!}\int \left[E - V(q)\right]^{(3N-5)/2} d^{3N}q.$$

Now the derivative with respect to E can be taken:

$$\frac{1}{T} = k_B\frac{1}{\Omega}\frac{d\Omega}{dE} = \frac{3N-5}{2}\frac{\int\left[E-V(q)\right]^{(3N-7)/2}d^{3N}q}{\int\left[E-V(q)\right]^{(3N-5)/2}d^{3N}q} = \frac{3N-5}{2}\left\langle\frac{1}{K}\right\rangle.$$

This expression contains a term $3N/(2K)$, which is intrinsic, i.e. it does not scale with the system size. This term is identical to the dominant term in (b). In addition, there is a correction term of order $1/N$ which is different from the correction term in (b). Therefore, the difference scales as $1/N$.

(d) The expectation value for the energy is given as

$$\langle E\rangle = \frac{\int E e^{-\beta E + S(E)/k_B}dE}{\int e^{-\beta E + S(E)/k_B}dE},$$

which, using our expansion, can be replaced by

$$\langle E\rangle = E^* + \frac{\int \Delta E\,\exp\left(-\frac{\alpha}{2}\Delta E^2 - \frac{\gamma}{6}\Delta E^3\right)d\Delta E}{\int \exp\left(-\frac{\alpha}{2}\Delta E^2 - \frac{\gamma}{6}\Delta E^3\right)d\Delta E}.$$

Evaluating the expectation value of the energy can now easily be done. We expand the third power in the exponent to linear order:

$$\langle E\rangle \approx E^* + \frac{\int_{-\infty}^{\infty} \Delta E\left[1-\frac{\gamma}{6}\Delta E^3\right] e^{-\frac{\alpha}{2}\Delta E^2}\, d\Delta E}{\int_{-\infty}^{\infty}\left[1-\frac{\gamma}{6}\Delta E^3\right] e^{-\frac{\alpha}{2}\Delta E^2}\, d\Delta E}.$$

Only the even powers in front of the Gaussians in the integrals survive, and working those out directly leads to

$$\langle E\rangle \approx E^* - \frac{3\gamma}{4\alpha^2}.$$

(e) The calculations seem cumbersome, but they can be easily done when writing the expression for Ω symbolically as

$$\Omega = \int (f(x)-E)^a\, dx,$$

where $a = (3N-5)/2$. Then we immediately find, using the definition

$$\left\langle K^y\right\rangle = \frac{\int (f(x)-E)^{a+y}\, dx}{\int (f(x)-E)^a\, dx}$$

that

$$-2k_{\mathrm{B}}\alpha = (a-1)\frac{d}{dE}\frac{\int (f(x)-E)^{a-1}\, dx}{\int (f(x)-E)^a\, dx} = \frac{d}{dE}\left\langle K^{-1}\right\rangle = (a-1)\left[(a-2)\left\langle K^{-2}\right\rangle - (a-1)\left\langle K^{-1}\right\rangle^2\right].$$

The expression for γ then proceeds straightforwardly along the same lines.

Space Mirrors

We deal with 3 reference frames:

S = frame of Enterprise,
S' = frame of mirror A,
S" = frame of mirror B.

These frames are connected via the Lorentz Transformation:

$$\begin{Bmatrix} ct' = \gamma\left(ct - \frac{V}{c}x\right) \\ x' = \gamma\left(x - \frac{V}{c}ct\right) \end{Bmatrix}$$. For S->S' this is $$\begin{Bmatrix} ct' = \frac{5}{3}\left(ct - \frac{4}{5}x\right) \\ x' = \frac{5}{3}\left(x - \frac{4}{5}ct\right) \end{Bmatrix}$$ and for S->S" $$\begin{Bmatrix} ct'' = \frac{5}{3}\left(ct + \frac{4}{5}x\right) \\ x'' = \frac{5}{3}\left(x + \frac{4}{5}ct\right) \end{Bmatrix}$$

In S the trajectory of mirror A is given by $x_A = \frac{V}{c}ct$ and for mirror B $x_B = -\frac{V}{c}ct$.

We define the following events:

E1: Light is send out in the positive x-direction at time ct=cT: $X_1^\mu = (cT, 0)$

Foton 4-momentum: $P_1^\mu = \left(\frac{hf_0}{c}, \frac{hf_0}{c}\right)$

E2: Light hits mirror A. The light was send out at cT, so its trajectory is $x_L = c(t - T)$

Thus light hits mirror: $x_L(t_2) = x_A(t_2) \rightarrow c(t_2 - T) = \frac{V}{c}ct_2 \rightarrow ct_2 = 5cT$

Thus in S: $X_2^\mu = (5cT, 4cT)$; in S' this is $X_2^{'\mu} = (3cT, 0)$

According to S' the light is described by

$$P_1^{'\mu} = \left(\frac{5}{3}\left[\frac{hf_0}{c} - \frac{4}{5}\frac{hf_0}{c}\right], \frac{5}{3}\left[\frac{hf_0}{c} - \frac{4}{5}\frac{hf_0}{c}\right]\right) = \left(\frac{1}{3}\frac{hf_0}{c}, \frac{1}{3}\frac{hf_0}{c}\right)$$

E3: Light is reflected at mirror A. According to S', this only means that the light propagates in the opposite direction. This in S' its 4-momentum is after the reflection:

$$P_3^{'\mu} = \left(\frac{1}{3}\frac{hf_0}{c}, -\frac{1}{3}\frac{hf_0}{c} \right)$$

Translating back to S: $P_3^{\mu} = \left(\frac{1}{9}\frac{hf_0}{c}, -\frac{1}{9}\frac{hf_0}{c} \right)$,

Doppler shifted as the mirror –according to S – is moving.

E4: Light has traveled to mirror B and hit it:

trajectory of light pulse from mirror A (according to S): $x_L = 4cT - c(t-5T)$;

trajectory of mirror B: $x_B = \frac{-4}{5}ct$.

Thus light will reach mirror B at $ct_4 = 45cT$. Thus in S: $X_4^{\mu} = (45cT, -36cT)$;

in S" this is $X_4^{'\mu} = (27cT, 0)$.

According to S" the light coming towards mirror B is described by

$$P_3^{''\mu} = \left(\frac{5}{3}\left[\frac{1}{9}\frac{hf_0}{c} + \frac{4}{5}\frac{-1}{9}\frac{hf_0}{c} \right], \frac{5}{3}\left[\frac{-1}{9}\frac{hf_0}{c} + \frac{4}{5}\frac{1}{9}\frac{hf_0}{c} \right] \right) = \left(\frac{1}{27}\frac{hf_0}{c}, \frac{-1}{27}\frac{hf_0}{c} \right)$$

After reflection at mirror B, this is according to S" the same pulse but now traveling in the positive direction: $P_4^{''\mu} = \left(\frac{1}{27}\frac{hf_0}{c}, \frac{1}{27}\frac{hf_0}{c} \right)$.

Translating back to S gives:

$$P_4^{\mu} = \frac{5}{3}\left(\frac{1}{27}\frac{hf_0}{c} - \frac{4}{5}\frac{1}{27}\frac{hf_0}{c}, \frac{1}{27}\frac{hf_0}{c} - \frac{4}{5}\frac{1}{27}\frac{hf_0}{c} \right) = \left(\frac{1}{81}\frac{hf_0}{c}, \frac{1}{81}\frac{hf_0}{c} \right)$$

This pulse reflects at $X_4^{\mu} = (45cT, -36cT)$ and will thus be detected by S at ct=81cT.

Shapiro spikes

(a) Follows by direct substitution of (3) into (2) and (1).

(b) After substitution of (5) into (4) we find for the supercurrent $I_s(t)$:

$$\begin{aligned} I_s(t) &= I_c \,\mathrm{Im}\left[\exp\left(i(\phi_0 + \omega_j t + z\sin(\alpha))\right)\right] \\ &= I_c \,\mathrm{Im}\left[\exp\left(i(\phi_0 + \omega_j t)\right)\exp(iz\sin(\alpha))\right] \end{aligned} \tag{9}$$

Due to the parity relation $J_k(z) = (-1)^k J_{-k}(z)$ of the Bessel functions, the components with odd k in the expansion (6) disappear from the first sum in (6), and the components with even k drop out of the second sum in this expansion. We then find for the supercurrent (9):

$$\begin{aligned} I_s(t) &= I_c \,\mathrm{Im}\left[\exp(i(\phi_0 + \omega_j t))\left(\sum_{k=-\infty}^{\infty} J_k(z)\cos(k\alpha) + i\sum_{k=-\infty}^{\infty} J_k(z)\sin(k\alpha)\right)\right] \\ &= I_c \,\mathrm{Im}\left[\exp(i(\phi_0 + \omega_j t))\sum_{k=-\infty}^{\infty}(-1)^k J_k(z)e^{-ik\alpha}\right] \\ &= I_c \,\mathrm{Im}\left[\sum_{k=-\infty}^{\infty}(-1)^k J_k(z)\,e^{i(\phi_0+\omega_j t)}\,e^{-ik\alpha}\right] \\ &= I_c \sum_{k=-\infty}^{\infty}(-1)^k J_k(z)\sin(\phi_0 + \omega_j t - k\alpha) \end{aligned} \tag{10}$$

Thus $f(k) \equiv (-1)^k J_k(z)$ and $x \equiv -k\alpha$.

(c) After substituting back and adding the shunt current V_0/R the total current I takes the form

$$I(t) = I_s(t) + \frac{V_0}{R} = I_c \sum_{k=-\infty}^{\infty}(-1)^k J_k\left(\frac{2eV_1}{\hbar\omega}\right)\sin\left(\phi_0 + \frac{2e}{\hbar}V_0 t - k\omega t\right) + \frac{V_0}{R}. \tag{11}$$

The dc part of the current is V_0/R unless

$$V_0 \equiv V_0^* = \frac{k\hbar\omega}{2e}, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{12}$$

The supercurrent then has the dc component

$$I_s = I_c(-1)^k J_k\left(\frac{2eV_1}{\hbar\omega}\right)\sin(\phi_0) \tag{13}$$

At the voltages V_0^* a sudden increase in supercurrent occurs, the so-called Shapiro spikes. Sketch: Linear increase with V_0 and sudden increases to the value (13) at the equally spaced voltages V_0^*.

(d) For small deviations around $V_0^* = \frac{k\hbar\omega}{2e}$ we write $\sin(\phi_0 + \frac{2e}{\hbar}V_0 t - k\omega t)$ in Eq. (11) as $\sin(\phi_0 + \epsilon t)$ with $\epsilon t \ll \phi_0 = \pi/2$ and use a Taylor expansion in ϵt. The supercurrent $I_s(t)$ in (11) then becomes

$$\begin{aligned} I_s(t) &= I_c \sum_{k=-\infty}^{\infty} (-1)^k J_k(\frac{2eV_1}{\hbar\omega}) \left(\sin(\phi_0)\cos(\epsilon t) + \cos(\phi_0)\sin(\epsilon t)\right) \\ &\overset{\phi_0=\pi/2}{=} I_c \sum_{k=-\infty}^{\infty} (-1)^k J_k(\frac{2eV_1}{\hbar\omega}) \cos(\epsilon t) \end{aligned}$$

The supercurrent now oscillates slowly with an amplitude $I_c \sum_{k=-\infty}^{\infty} (-1)^k J_k(\frac{2eV_1}{\hbar\omega})$.

Cooperative binding in biological systems

A) We first write the energy of all the microstates $b_1 b_2 b_3 b_4$ and group them:

0000=-4ϵ

1000=$-\mu$
0100=$-\mu$
0010=$-\mu$
0001=$-\mu$

1100=-2μ
0110=-2μ
0011=-2μ
1001=-2μ

1010=$-2\mu+4\epsilon$
0101=$-2\mu+4\epsilon$

1110=-3μ
1011=-3μ
1101=-3μ
0111=-3μ

1111=$-4\epsilon-4\mu$

The partition function is therefore:

$$Z = e^{4\epsilon\beta} + 4e^{\mu\beta} + 2e^{2(\mu-2\epsilon)\beta} + 4e^{2\mu\beta} + 4e^{3\mu\beta} + e^{4(\mu+\epsilon)\beta}$$

B) When $\mu = 0$, the partition function simplifies to:

$$Z = e^{4\epsilon\beta} + 4 + 2e^{-2(2\epsilon)\beta} + 4 + 4 + e^{4(\epsilon)\beta}$$
$$= 2e^{4\epsilon\beta} + 2e^{-4\epsilon\beta} + 12$$

If we let L=number of bound ligands, then:

$$\langle L\rangle = \frac{(0)\cdot e^{4\epsilon\beta} + (1)\cdot 4e^{\mu\beta} + (2)\cdot 2e^{2(\mu-2\epsilon)\beta} + (2)\cdot 4e^{2\mu\beta} + (3)\cdot 4e^{3\mu\beta} + (4)\cdot e^{4(\mu+\epsilon)\beta}}{Z}$$
$$= \frac{0 + 4 + 4e^{2(-2\epsilon)\beta} + 8 + 12 + 4e^{4(\epsilon)\beta}}{Z}$$
$$= \frac{4e^{4\epsilon\beta} + 4e^{-4\epsilon\beta} + 24}{Z} = 2$$

The probability of exactly 2 proteins are bound is:

$$Pr(L=2) = \frac{2e^{2(-2\epsilon)\beta} + 4}{Z} = \frac{2e^{-4\epsilon\beta} + 4}{2e^{4\epsilon\beta} + 2e^{-4\epsilon\beta} + 12}$$

$$= \frac{1}{1 + \frac{e^{4\epsilon\beta} + 4}{e^{-4\epsilon\beta} + 2}}$$

From this equation we can see that an increasing value in ϵ causes the probability to asymptotically approach zero. Therefore the cooperative system tends to exist as either a mostly bound or mostly unbound state rather than the half-bound state.

C) The partition function simplifies to:

$$Z \approx e^{4\epsilon\beta} + 4e^{\mu\beta}$$

The average energy of the complex is:

$$\langle U \rangle = -\frac{\partial}{\partial\beta}\ln(Z) = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = -\frac{4\epsilon e^{4\epsilon\beta} + 4\mu e^{\mu\beta}}{e^{4\epsilon\beta} + 4e^{\mu\beta}}$$
$$\approx 4\epsilon$$

The probability of any ligands being bound is:

$$Pr(L > 0) \approx \frac{(0)e^{4\epsilon\beta} + (1)4e^{\mu\beta}}{Z} = \frac{4e^{\mu\beta}}{e^{4\epsilon\beta} + 4e^{\mu\beta}}$$
$$\approx \frac{4e^{\mu\beta}}{e^{4\epsilon\beta}} = 4e^{(\mu-4\epsilon)\beta}$$

From this equation we can see that an increasing value in ϵ causes the probability to asymptotically approach zero. Therefore the cooperative system tends to show less binding at dilute ligand concentrations.

D) There are several ways to solve this problem. First we note that rather than keeping track of the bound state of each protein, we can just mark whether the state "flips"=f between neighboring proteins. The number of flips must be even because the complex is a ring. Also, for any given configuration of flips, we can specify two states based on the state of the first protein in the complex. This lets as write the partition function as a relatively compact sum:

$$Z = 2\sum_{f=0}^{N/2} \binom{N}{2f} e^{-2f\epsilon\beta} e^{(N-2f)\epsilon\beta}$$

We can test this formula for N=4 and see that it matches our earlier answer. Next we rewrite Z in terms of $n = 2f$ to be:

$$Z = \sum_{n=0}^{N} \binom{N}{n} e^{-n\epsilon\beta} e^{(N-n)\epsilon\beta} + \sum_{n=0}^{N} (-1)^n \binom{N}{n} e^{-n\epsilon\beta} e^{(N-n)\epsilon\beta}$$

Note that all the odd values of *n* cancel, making this equivalent to the previous formulation. Finally, we re-write these sums as:

$$Z = \left(e^{-\epsilon\beta} + e^{-n\epsilon\beta}\right)^N + \left(e^{-\epsilon\beta} - e^{-n\epsilon\beta}\right)^N$$
$$Z = (2\cosh(\epsilon\beta))^N + (2\sinh(\epsilon\beta))^N$$